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THE PASSAGE OF ENERGETIC CHARGED PARTICLES THROUGH INTERPLANETARY SPACE*

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Abstract—The passage of cosmic ray particles and energetic solar particles through interplanetary space is illustrated with a number of idealized examples. The formal examples are worked out from the condition that energetic particles in interplanetary space random walk in the irregularities in the large-scale interplanetary magnetic field. The irregularities move with approximately the velocity of the solar wind. The classical probability distribution is describable by a Fokker–Planck equation. A general expression for the particle diffusion coefficient κ_{ij} is worked out, including both scattering in magnetic irregularities and systematic pressure drifts. Magnetometer data from Explorer XVIII is presented to show the close average adherence of the quiet-day interplanetary magnetic field to the theoretical spiral angle, and to show the tendency for particles to move more freely along the field than across, $\kappa_{\parallel} > \kappa_{\perp}$. The observed fields show that the diffusion coefficient is of the order of 10^{21}–10^{22} cm^2/sec, as had been estimated from earlier cosmic ray studies. A middle value of 3×10^{21} cm^2/sec suggests a cosmic ray density gradient of about 10 per cent per a.u. across the orbit of Earth. Direct observations of the interplanetary magnetic field afford the possibility for quantitative estimate of κ_{ij} as a function of particle energy.

The first example to be considered is isotropic diffusion in a spherical region $r < R$ with uniform radial wind velocity v for the purpose of illustrating the general nature and duration of the passage of a cosmic ray particle through the solar system. It is shown that the cosmic ray density reduction is of the order of $\exp(-vR/\kappa)$, and, hence, that during the years of solar activity vR/κ is not less than about 1 for protons of one BeV or so. It follows from this that the galactic cosmic ray particles will generally have spent several days in the solar system by the time they are observed. During this time they are in the expanding magnetic fields carried in the solar wind and are adiabatically decelerated, losing 15 per cent or more of their energy by the time they are observed. The energy distribution is shown for particles starting all with the same energy T_0 from interstellar space. The incoming probability wave of a single particle is computed as a function of time, showing how the particle is swept back by the wind.

The converse problem of energetic solar particles is illustrated. The solar particles may typically lose half their initial energy before escaping into interstellar space. The outward motion of the wind displaces their probability distribution outward so that ultimately the maximum solar particle intensity may lie beyond the orbit of Earth. The outward motion of the wind steepens the decline of the solar particle intensity.

The steady-state cosmic ray intensity is calculated throughout the spherical region $r < R$ supposing a uniform cosmic ray density N_0 to obtain in interstellar space. The calculation is carried out for isotropic κ_{ij}, which would obtain if the magnetic irregularities were of large amplitude and of a scale not exceeding the radius of gyration of the cosmic ray particles, and for anisotropic κ_{ij} with $\kappa_{\parallel} \gg \kappa_{\perp}$, which obtains when the field is relatively smooth. (The observations at sunspot minimum suggest $\kappa_{\parallel} \gg \kappa_{\perp}$ at the orbit of Earth.) The particles diffuse only along the spiral lines of force when $\kappa_{\parallel} \gg \kappa_{\perp}$, so their path in and out of the solar system is much longer than when κ_{ij} is isotropic. The result is a much greater reduction of the cosmic ray intensity for a given $vR/|\kappa_{ij}|$.

There is no direct observational information on κ_{ij} beyond the orbit of Earth, where the intensity reduction takes place. Indirect information is available, however. There is the fact that the intensity of energetic solar particles at Earth often decays as t^{-g} with $g = 1{\cdot}5$–$2{\cdot}0$. It is shown that in order for this to follow, it is necessary that $|\kappa_{ij}| \propto r^s$ with $s = 0{\cdot}0$–$0{\cdot}5$ if κ_{ij} is isotropic, and $s = 2{\cdot}0$–$2{\cdot}5$ if $\kappa_{\parallel} \gg \kappa_{\perp}$. That is to say, if κ_{ij} should continue to be as anisotropic beyond Earth as it is observed to be near Earth, then the diffusion must increase rapidly with distance from the Sun. These qualitative features should be easily detectable with particle, field, and plasma observations beyond the orbit of Earth.

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1. INTRODUCTION

(a) General remarks

Galactic cosmic ray particles reaching Earth must penetrate the interplanetary magnetic fields lying outside the orbit of Earth. Energetic particles from the Sun must penetrate the same fields to escape. There is no direct observational information on the fields beyond Earth, but a sufficient number of inferences can be made to permit a qualitative picture of the passage of energetic particles. This paper explores the general nature of the passage of energetic particles through the inferred quiet-day interplanetary magnetic fields.

Galactic cosmic ray particles penetrate into the solar system against the outward sweep of the magnetic fields carried in the solar wind, leading to a reduced cosmic ray intensity here in the solar system. The theoretical prediction of the general nature and extent of the interplanetary magnetic fields[1–5] made it possible to construct the qualitative features of the reduction[3,6] and to connect it with the inverse problem of the escape of energetic solar particles into interstellar space[5]. The present paper looks farther into the general properties of the propagation of energetic charged particles through interplanetary space, considering transit time, energy loss, and the outward convection of solar particles, which hitherto have been ignored. The various qualitative conclusions of the paper are based on exact calculations of idealized models of the interplanetary field which contain the essential physical features of the actual fields. The purpose is to point out and to illustrate the essential physical behavior of energetic charged particles in interplanetary space.

It is shown, for instance, that most of the cosmic ray particles incident on the solar wind from interstellar space are reflected immediately back into space, with a small energy gain as a consequence of their head-on collision with the outward moving fields in the solar wind. Only a very small fraction of the incident particles diffuse into the solar wind, where they remain for a considerable period of time (days or weeks) and lose a significant fraction of their energy to the expanding interplanetary fields before returning to interstellar space. Energetic particles from the Sun are similarly decelerated before escaping into interstellar space. The outward convection of solar particles hastens the steepening of the initial $t^{-\alpha}$ decline of the particle density after a flare. In contrast to this, a low scattering rate tends to decrease α.

But before going into the formal calculations which demonstrate these effects, we digress long enough to consider the present state of knowledge of the interplanetary fields on which the formal calculations are based.

(b) Interplanetary magnetic fields

First of all, the theory of the solar wind predicted in the basic quiet-day pattern of the interplanetary magnetic field is everywhere (rather than only in isolated streams) an Archimedes spiral with the Sun at its origin. The theory predicted a quiet-day field of the order of 3×10^{-5} G at the orbit of the Earth, based on a mean field of one gauss at the Sun. Second, irregularities on a scale of 10^{5}–10^{8} km in the spiral pattern were expected from theoretical considerations[1,5]. Irregularities over extensive regions of space were inferred from cosmic ray observations[7–12]. Recently the interplanetary fields have been observed directly, verifying the general spiral pattern[13] and giving direct quantitative information on the nature of the irregularities[13,14].

The observations of Ness *et al.* are sufficiently comprehensive and important for the present discussion of the quiet-day field to display them in Fig. 1 from orbits 11 and 15

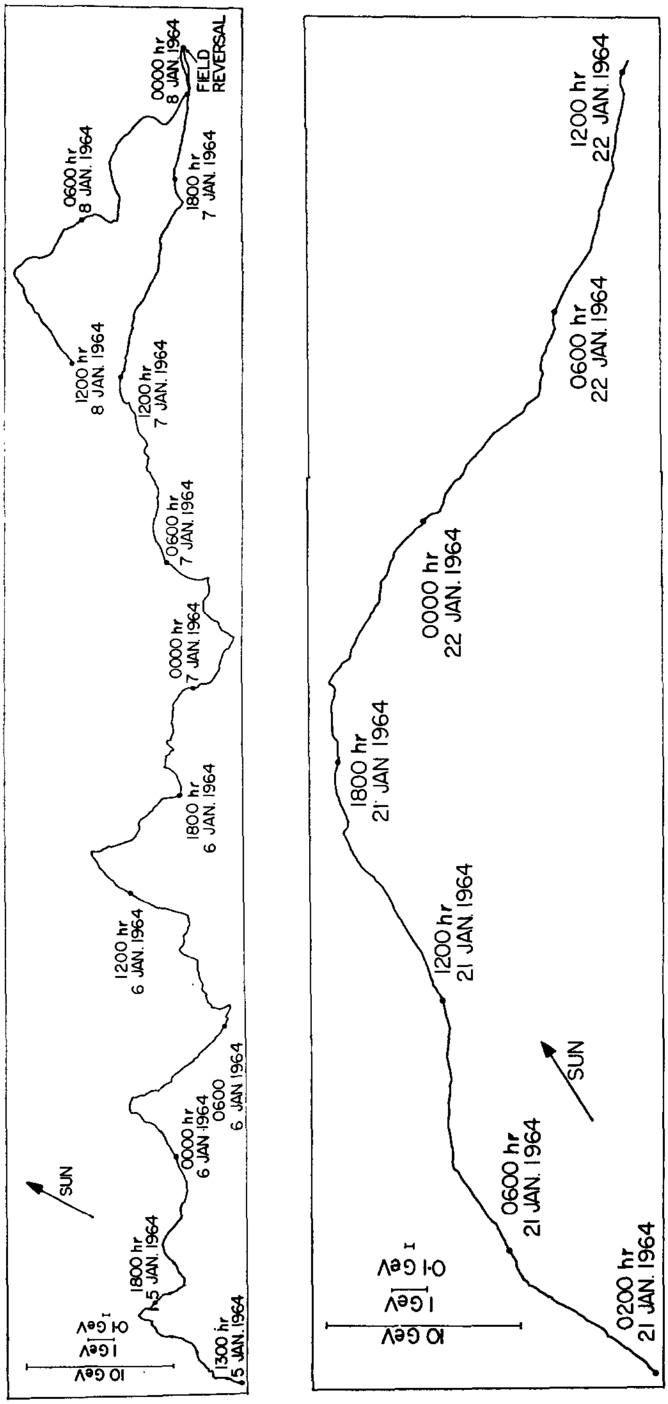

FIG. 1. THE DIRECTION OF THE INTERPLANETARY MAGNETIC FIELD IN THE PLANE OF THE ECLIPTIC AS OBSERVED BY NESS *et al.*[13] FROM EXPLORER XVIII IS PLOTTED TO SHOW THE UNDERLYING SPIRAL PATTERN. THE DIRECTION TO THE SUN IS INDICATED BY THE ARROW AND REMAINS FIXED THROUGHOUT EACH PLOT.

of Explorer XVIII. For detailed discussion of the magnetometer and the data the reader is referred to Ness *et al.*[13] The variations of the interplanetary magnetic field recorded by Explorer XVIII are spatial irregularities transported rigidly in the solar wind blowing by the vehicle (the Alfvén speed, with which the irregularities move relative to the wind, is only about 50 km/sec). The plots shown in Fig. 1 were drawn by causing the pen to progress with constant speed across the paper (representing the plane of the ecliptic) in the direction of the interplanetary field at each instant of time. Thus, had the wind been transporting the field in the direction of the magnetic field, the line in Fig. 1 would be an approximate map of a line of force of the field. Instead, the line has merely the same general characteristic direction and the same general irregularities as the actual lines of force. Time is indicated along each line*. The scale of the field is indicated by the middle segment, labelled 10^9 eV, which represents 1×10^6 km on the assumption that the wind velocity is a uniform 400 km/sec. The sharp reversal of the line in Fig. 1(a) at about 2200, 7 January 1964 represents passage of Explorer XVIII into the field from a region of opposite polarity on the Sun.

In looking at Fig. 1 there are several features which stand out. First of all, there is the long narrow path followed by the field across Fig. 1, indicating the close average confinement of the field to the theoretical spiral angle, already pointed out by Ness *et al.*[13] Second there is the continual presence of irregularities of all scales above the 300 sec ($1{\cdot}2 \times 10^5$ km) intervals between data samples. Third, there is the striking tendency in Fig. 1(a) for the field to bend sharply through one radian or more at intervals of the order of several million km, say 2–10×10^6 km. The field between the bends curves relatively gently, except for a few sharp wiggles of small scale ($<10^6$ km). In Fig. 1(b) the field has the same general characteristics as in Fig. 1(a) except that the sharp bends every several million km are missing.

In a typical interplanetary field of 5×10^{-5} G, the radius of gyration of a proton moving perpendicular to the field is 0·28, 1·0 and 6·6 million km at 10^8, 10^9 and 10^{10} eV, respectively. These lengths are layed out in Fig. 1 for direct comparison with the observed scale of the irregularities in the field: It is readily seen that the 10^{10} eV proton has a radius of gyration comparable to the distance between the sharp bends in the field in Fig. 1(a); the 10^9 eV proton has a radius of gyration small compared to the distance between bends but comparable to the radius of curvature of many of the bends; the 10^8 eV proton has a radius which is small compared to any feature of the large bends but which is comparable to some of the occasional small-scale fluctuations.

(*c*) *Particle motions in the interplanetary fields*

Calculations of the motion of a charged particle in a large-scale field containing small-scale irregularities, such as in Fig. 1, shows that a particle is most effectively scattered by irregularities which have a scale comparable to the radius of gyration of the particle[15]. Particles of higher rigidity pass through each irregularity with but little deflection. The calculations show further that, if the irregularity does not leave a net final displacement in the large-scale line of force, the net deflection is still further reduced. This is because with no final displacement of the lines of force the deflection of the particle over the first displacement of the line of force is almost exactly cancelled by the deflection in the return displacement of the line of force. It is of particular importance, then, to decide whether

* There is a gap of a couple of hours in the data for orbit 11 at about 1800 hours, 6 January 1964. The gap is not included in the figure since it is not known in what direction to plot the gap.

the magnetic irregularities each represent a net shear or displacement of the lines of force, or whether each displacement is followed immediately (within one radius of gyration) by a restoring displacement. It is interesting to note that in the field illustrated in Fig. 1(a) the sharp bends represent a net displacement of the field so far as 10^9 eV particles are concerned. The return bend is removed more than one radius of gyration. On the other hand the radius of gyration of a 10^{10} eV particle is comparable to the distance between bends, so for these higher energy particles the net irregularity has but little net displacement of the line of force.

Particles whose radius of gyration is small compared to the scale of the field irregularities pass smoothly through the irregularities, with their motion described by the guiding center approximation. The particles may be deflected from regions of strong field along the lines of force, but otherwise pass through freely with little or no change in their magnetic moment. An isotropic particle distribution where the particles are fed onto the line of force means a uniform particle density everywhere along the line of force under ordinary circumstances*. Thus particles with sufficiently small magnetic rigidity may perhaps penetrate into the solar system more easily than particles of higher rigidity. Low energy electrons, with their relativistic speed and low rigidity are particularly likely candidates for this. A 10 MeV electron in a field of 5×10^{-5} G has a radius of gyration of only 6×10^3 km.

From the theoretical behaviour of a charged particle in an irregular magnetic field [15,16] it follows from the observations that

(a) for the field observed during orbit 11 cosmic ray particles over the entire range 10^8–10^{10} eV are scattered by the irregularities in the spiral field. For orbit 15 protons of 10^{10} eV are scattered considerably less than particles at 10^9 eV and below. The mean free path along the field between scattering lies in the range $1–10 \times 10^6$ km (except for 10^{10} eV protons in orbit 15), yielding diffusion coefficients of 10^{21}–10^{22} cm^2/sec, respectively. We have no way of inferring from this the mean free path and diffusion coefficient far beyond the orbit of Earth, but it is interesting to note that these numbers are in agreement with the earlier order of magnitude estimates from cosmic ray observations[5,8,10,17,18].

(b) the scattering of 10^9 eV protons is somewhat more effective than the scattering of 10^{10} eV protons in Fig. 1(a) because the 10^9 eV proton experiences a net change in the direction of the field at each sharp bend, whereas the 10^{10} eV particle "sees" across the interval between bends, so that for it the field fluctuates but experiences no *net* change in direction[15]. For this reason the particles below 10^{10} eV have a mean free path along the field which is comparable to the distance between sharp bends, and hence is much larger than their mean free path (radius of gyration) across the field. The same is true at 10^{10} eV but probably to a lesser degree. In Fig. 1(b) the difference in the mean free paths parallel and perpendicular to the field is even more striking.

The tendency for particles to move along the magnetic lines of force of the spiral more freely than across, as pointed out in (b), is an essential part of the theory[16] of the diurnal variation of the cosmic ray intensity at Earth. Some of the additional effects of such anisotropy are illustrated in section 4.

The steady change of the spiral angle exhibited in Fig. 1(b) is particularly interesting, since presumably it results from variations of the wind velocity around the Sun and perhaps with time[3,5]. Presumably individual instances can be understood in detail as more magnetic and wind information become available.

Simpson[12] has pointed out the different time behavior of the cosmic ray density at different proton energies over the 11-year cycle of solar activity. We would suppose that the different behavior is attributable to variations in the qualitative features of the magnetic irregularities, mentioned in (a) and (b) as much as to variations in average field strength, wind velocity, and extent of the solar wind into space.

* The exception is when a temporary constriction in the line of force chokes off the particle flow, such as occurs in the field through a blast wave from a solar flare[3,5].

It is interesting to note from (a) and (b) that a typical mean free path L of 5×10^6 km along the magnetic line of force yields a diffusion coefficient $\kappa_{\parallel} \cong 1/3\, cL$ of about 5×10^{21} cm^2/sec. From this one would expect a variation in the cosmic ray intensity of the order of 15 per cent per a.u. as a consequence of the outward motion of the fields, which order of magnitude is in agreement with the low value set by observation[19,20]. A factor of e arises over a distance of about 7 a.u., in agreement with the very rough estimates made earlier[5,6].

The time $l^2/4\kappa$ to diffuse a distance $l = 1$ a.u., as in the arrival of energetic particles from a flare, is 10^4 sec or about three hours. Considerably longer periods of time are required for diffusion across the lines of force (section (b) above). This would seem to account for much, if not all, of the observed delay of arrival of solar particles at Earth. And with (b) it suggests that the example worked out elsewhere (Parker[5], p. 228) for the arrival of energetic solar particles from a flare on the back side of the Sun might be extended to include a finite anisotropic diffusion coefficient at all radial distances from the Sun, instead of an isotropic coefficient beyond a certain distance with free radial passage closer to the Sun.

2. FOKKER–PLANCK EQUATION

(a) *General form of the Fokker-Planck equation*

The feature of the interplanetary magnetic field which determines the nature of the propagation of energetic particles is the general presence of small-scale irregularities in the field. The irregularities appear with dimensions of 10^5–10^7 km, which are comparable to the radius of gyration of typical cosmic ray particles, but which are small compared to the overall dimensions of interplanetary space. The irregularities scatter, or reflect, the energetic particles back and forth along the lines of force of the large-scale field, so that there is no tendency for the particles to move systematically in either direction in the frame of reference of the irregularities. Viewed from the large scale, then, the effect of the magnetic irregularities is to cause the cosmic ray particles to random walk in the frame of reference of the magnetic irregularities. If the scattering is infrequent (compared to the cyclotron frequency), then the particles random walk back and forth along a line of force with little diffusion across the lines of force. The particle motion is describable by the well known guiding center approximation between scatterings. If, on the other hand, significant scattering occurs as frequently as once each cyclotron period, then diffusion across lines of force becomes important too. It is evident that the diffusion coefficient describing this random walk is a tensor quantity κ_{ij} with a larger value parallel than perpendicular to the large-scale field.

The random walk of the cosmic ray particles is a Markhoff process, describable by a Fokker–Planck equation (see formal discussion in Chandrasekhar[21]). To describe the random walk, introduce the classical probability distribution $W(x_i, t)$ of the particle. Denote the diffusion coefficient by κ_{ij}, which is defined such that $-\kappa_{ij}\partial W/\partial x_j$ is the particle flux in the frame of reference moving with the magnetic irregularities producing the scattering. The magnetic irregularities move with the solar wind, of course, with velocity v_i, so that in the fixed frame of reference there is an additional particle flux $v_i W$ of convective origin. The divergence of the total particle flux gives the accumulation at a point, yielding the Fokker–Planck or diffusion equation

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial x_i}(Wv_i) - \frac{\partial}{\partial x_i}\left(\kappa_{ij}\frac{\partial W}{\partial x_j}\right) = 0 \qquad (1)$$

for the particle distribution $W(x_i, t)$[5,6,15,16].

Now while the energetic particle is riding along with the fields in the wind, the magnetic fields in which the particle is moving are expanding because of the radial divergence of the wind. The energetic particle is cooled adiabatically*, so that its momentum p declines as

$$\frac{1}{p}\frac{dp}{dt} = -\frac{1}{3}\frac{\partial v_i}{\partial x_i}$$

and its kinetic energy T declines as

$$\frac{1}{T}\frac{dT}{dt} = -\frac{n(T)}{3}\frac{\partial v_i}{\partial x_i} \tag{2}$$

where $n(T) = 2$ for nonrelativistic particles and $n(T) = 1$ for extreme relativistic particles. We shall work with the particle distribution over T, plotting the results for the two cases $n = 1, 2$. To obtain the distribution over p, it is necessary only to replace T by p and put $n = 1$. For a radial wind of constant speed v, the divergence $\partial v_i/\partial x_i$ is equal to $2v/r$. If $U(x_i, T, t)$ represents the probability distribution over kinetic energy, so that

$$W(x_i, t) = \int_0^\infty dT\, U(x_i, T, t),$$

then the Fokker–Planck equation for U is

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x_i}(Uv_i) + \frac{\partial}{\partial T}\left(U\frac{dT}{dt}\right) - \frac{\partial}{\partial x_i}\left(\kappa_{ij}\frac{\partial U}{\partial x_j}\right) = 0 \tag{3}$$

As we noted in the Introduction, we shall be concerned with the time of passage of an energetic particle into and/or out of the solar system, with the energy which the particle may lose, and with the tendency for the particle to diffuse more readily along the spiral field than across. The purpose is to point out and illustrate the effects, so that eventually when increasing observational information permits, they can be studied quantitatively. It is sufficient, therefore, in this first study to compute the time of passage and the energy loss with an isotropic diffusion coefficient, in which case (2) reduces to

$$\frac{\partial U}{\partial t} + \frac{v}{r^2}\frac{\partial}{\partial r}(r^2U) - \frac{2v}{3r}\frac{\partial}{\partial T}[UTn(T)] - \frac{1}{r^2}\frac{\partial}{\partial r}\left(\kappa r^2\frac{\partial U}{\partial r}\right) = 0 \tag{4}$$

for radial diffusion and to treat the effects of anisotropy separately.

(b) The general diffusion coefficient

In treating the effects of anisotropy in a spiral field etc., the anisotropic character of κ_{ij} must be retained. To compute the form of κ_{ij} let L be the length of the step the particle makes along the magnetic field B_i. Let ν represent the number of steps taken in unit time. Then if the particle velocity is w_i we would expect that $\nu = w_{\parallel}/L$, approximately, where $w_{\parallel}$ denotes the component of w_i along B_i. The diffusion coefficient along the field is

$$\kappa_{\parallel} \cong \nu L^2$$

The radius of gyration of the particle across the field is $S = w_{\perp}/\Omega$ where Ω is the cyclotron

* Neglecting possible Fermi acceleration (discussed in Appendix 6) which *seems* to be negligible.

frequency of the particle in the field. Hence, the diffusion coefficient across the field is

$$\kappa_\perp \cong \nu S^2$$

if the particle is closely tied to the line of force, $\nu \ll \Omega$, i.e. if $S \ll L$. If on the other hand $L \ll S$, i.e. if the particle is scattered many times in one cyclotron period ($\nu \gg \Omega$), then $\kappa_\perp \cong \kappa_\parallel$. A sufficient tensor representation of these effects is

$$\kappa_{ij} \cong \nu L^2 \left(\frac{\nu^2\delta_{ij} + \Omega_i\Omega_j}{\nu^2 + \Omega^2}\right) \tag{5}$$

where

$$\Omega_i \equiv \frac{qB_i}{Mc} \tag{6}$$

and q and M are the total charge and mass of the particle. This expression for κ_{ij} adequately describes the scattering of the guiding center of the particle. But there may be a net streaming as a consequence of a pressure gradient in the particle density, or a drift of the guiding centers. To include these we note[22] that the net particle streaming u_i at a point in a large-scale field is

$$u_i = \frac{c}{NqB^2}\,\epsilon_{ijk}\,B_j\left[\frac{\partial p_\perp}{\partial x_k} + (p_\parallel - p_\perp)\frac{B_n}{B^2}\frac{\partial B_k}{\partial x_n} + NM\frac{dv_k}{dt}\right] \tag{7}$$

for particles of mass M and charge q, where $p_\parallel$ and $p_\perp$ denote the particle pressures parallel and perpendicular to B_i, dv_k/dt is the acceleration of the solar wind and ϵ_{ijk} is the usual permutation tensor, equal to ± 1 according as ijk is an even or odd permutation of 1, 2, 3 and zero otherwise. The acceleration term dv_k/dt can be neglected under most circumstances, as can the cosmic ray anisotropy* $p_\parallel - p_\perp$ for the present purposes (see discussion[6]). It is sufficient for present purposes to neglect the changes in particle energy $\frac{1}{2}Mw^2$, so that $p_\perp \equiv \frac{1}{3}NMw^2$ varies only with N. It follows that the streaming u_i can be written

$$Nu_i \cong \frac{w^2}{3\Omega^2}\,\epsilon_{ijk}\Omega_j\,\frac{\partial N}{\partial x_k} \tag{8}$$

Then since $\nu^2L^2 = w^2/3$, we may represent this pressure drift by the artifice of the diffusion coefficient

$$\kappa_{ij} = +\frac{\nu^2L^2}{\Omega^2}\,\epsilon_{ijk}\Omega_k \tag{9}$$

Combining (5) and (9) we have altogether

$$\kappa_{ij} \cong \frac{\nu L^2}{\nu^2 + \Omega^2}\,[\nu^2\delta_{ij} + \Omega_i\Omega_j + \nu\epsilon_{ijk}\Omega_k] \tag{10}$$

as a sufficient approximation to the diffusion coefficient for particles with cyclotron frequency Ω and random walk frequency ν. Each of the terms in (10) is correct for those values of ν/Ω for which the term is non-negligible. The expression is approximately correct

* There are brief periods where the anisotropy[8,10,17] is sufficiently large that it should not be neglected.

for both non-relativisitc and extreme relativistic particles. We employ (10) for the discussion of the passage of energetic particles through interplanetary space.

3. PASSAGE OF COSMIC RAY PARTICLES INTO THE SOLAR SYSTEM

(a) *The computational model*

The physical model employed in the calculations to illustrate the time of passage and energy loss to cosmic rays and solar particles may be fairly simple, since neither effect depends critically on the details. We shall ignore all the interesting complications that may occur in the outer regions of the solar wind[4,11] and shall suppose that the solar wind blows radially with constant velocity v and sweeps back the cosmic rays to a distance R, beyond which is free interstellar space where the cosmic ray density is isotropic and uniform with N_0 particles/cm^3. The discussion will be limited to non-relativistic particles and extreme relativistic particles so that $n(T)$ may be taken as a constant, with a value $n = 2$ and $n = 1$ respectively.

To obtain an idea of the order of magnitude of the time of passage and the energy loss, let κ represent the cosmic ray diffusion coefficient, which for the present discussion we take to be isotropic and uniform out to $r = R$. It is readily shown[5,6] that the cosmic ray density here in the inner solar system is reduced to $N_0 \exp(-Rv/\kappa)$ by the outward motion of the wind. The observed amplitude of the 11-year variation of the cosmic ray intensity shows that Rv/κ is probably of the general order of unity. In a time t a cosmic ray particle diffuses a distance $(4\kappa t)^{1/2}$, so that to arrive in the inner solar system from $r = R$ requires a time

$$t = \frac{R}{4v}\frac{Rv}{\kappa} = O\left(\frac{R}{4v}\right), \tag{11}$$

which is just one fourth the time it takes the wind to reach $r = R$. The solar wind velocity is, say, 400 km/sec[23] so that if R is small as 5 a.u. the time t is 5 days, in order of magnitude. There are suggestions[12] that R may perhaps be as large as 40 a.u., giving t of the order of a month. These diffusion times are to be compared with the one hour and the five hours, respectively, in which a cosmic ray particle would traverse the same distances if there were no irregularities in the magnetic fields.

The characteristic time t_E in which the kinetic energy of a particle falls by a factor of e is readily shown from (2) to be

$$\frac{1}{t_E} \equiv -\frac{1}{T}\frac{dT}{dt}$$

$$= \frac{2n(T)}{3}\frac{v}{r}$$

in a uniform radial solar wind with velocity v. Assigning R as the characteristic value of r, and using (11) gives

$$t_E \cong \frac{n(T)}{6} t \tag{12}$$

So by the time a particle arrives here its energy is the fraction $\exp(-t/t_E) = \exp(-n/6)$ of the energy which it had in interstellar space. For a non-relativistic particle $n = 2$ and the energy is 0·7 the initial energy; for an extreme relativistic particle the energy is 0·85. These rough estimates are conservative, as will be seen later when the complete calculation is carried out from (4). Both the individual particle energy loss and the depression of the particle density contribute to the observed reduced particle intensity. The energy loss to energetic solar particles diffusing out through the solar wind into interstellar space is considerably greater than the energy loss to a galactic particle first arriving at Earth because the solar particles spend a larger fraction of their time at small r.

The formal calculations in this section will be carried out with the assumption that $\nu \gg \Omega$ so that (10) reduces to the isotropic tensor* $\delta_{ij}\nu L^2$. It will also be assumed that κ_{ij} is independent of particle energy T and uniform inside $r = R$. The particle diffusion over azimuthal angle ϕ and polar angle θ is not of prime interest† so the computational model used here, including the cosmic ray intensity at $r = R$, has spherical symmetry. So the calculations will be restricted to the radial particle distributions $U(r, t, T)$ and $W(r, t)$, omitting dependence on θ and ϕ. The restriction to radial dependence may be achieved in two ways. We may either introduce the particles in a spherically symmetric manner so that $U(r, \theta, \phi, t, T)$ is automatically independent of θ and ϕ, or if we like, we may introduce a single particle at a given point and define $U(r, t, T)$ as the integral of the resulting $U(r, \theta, \phi, t, T)$ over θ and ϕ,

$$U(r, t, T) = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \, U(r, \theta, \phi, t, T)$$

In either case the distribution $U(r, t, T)$ satisfies (4). The same is true for $W(r, t)$.

(b) Time of passage

Suppose that a cosmic ray particle from interstellar space crosses $r = R$ into the solar system and is scattered by an irregularity (at time $t = 0$) in the magnetic field after having penetrated a radial distance h. Obviously h is of the order of the scattering length L, on which we will have more to say later. At the instant of the scattering the probability distribution of the particle is a Dirac delta function in space, located at the point of scattering

$$W(r, 0) = \frac{\delta[r - (R - h)]}{4\pi(R - h)}, \tag{13}$$

$$4\pi \int_0^R dr\, r^2 W(r, 0) = 1 \tag{14}$$

Subsequently $W(r, t)$ is determined by the Fokker–Planck equation

$$\frac{\partial W}{\partial t} + \frac{v}{r^2}\frac{\partial}{\partial r}(r^2 W) = \frac{\kappa}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial W}{\partial r}\right) \tag{15}$$

The particle is assumed to escape freely back into interstellar space when it returns again to $r = R$, so there is the boundary condition

$$W(R, t) = 0 \tag{16}$$

* The formal computations will be valid even if κ_{ij} is anisotropic provided only that κ_{ij} is symmetric and its principal axes lie along the coordinate directions.

† Theoretical examples of diffusion over ϕ and θ may be found elsewhere[5].

The formal solution of (15) subject to the initial condition (13) and the boundary condition (16) is worked out in Appendix 1. In Appendix 2 the same problem is considered in a one dimensional space because the result can be expressed in closed analytical form. The probability wave of the particle diffusing into the solar system against the outward sweep of the wind is shown in Fig. 2 for the three cases of no wind $Rv/\kappa = 0$, a moderate wind

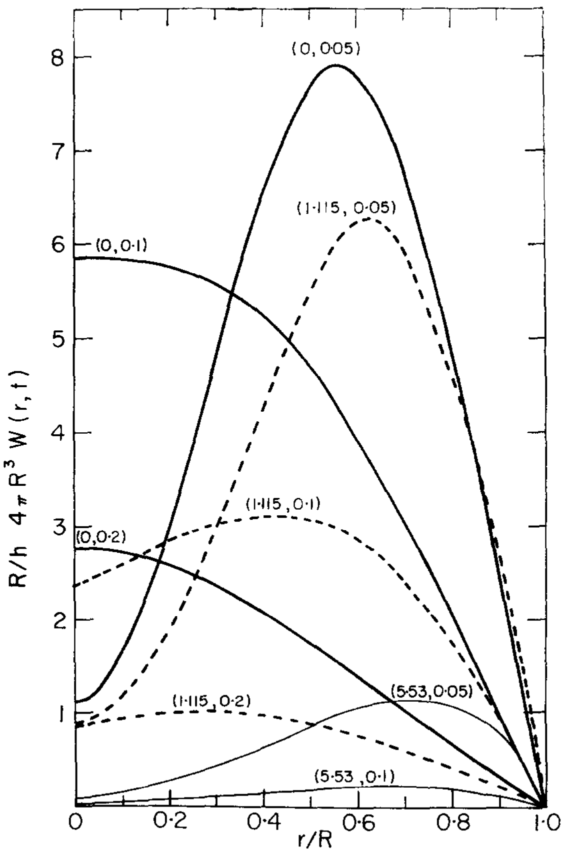

FIG. 2. THE PROBABILITY DISTRIBUTION $W(r, t)$ IS PLOTTED FOR THE TIMES $\kappa t/R^2 = 0{\cdot}05$, $0{\cdot}1$ AND $0{\cdot}2$. THE HEAVY LINES ARE FOR NO SOLAR WIND, $v = 0$: THE BROKEN LINES ARE FOR A MODERATE WIND $Rv/\kappa = 1{\cdot}115$: THE LIGHT LINES ARE FOR A VERY STRONG WIND, $Rv/\kappa = 5{\cdot}53$. THE VALUES OF Rv/κ AND $\kappa t/R^2$ ARE SHOWN IN THE PARENTHESES ASSOCIATED WITH EACH CURVE.

$Rv/\kappa = 1{\cdot}115$, and a strong wind $Rv/\kappa = 5{\cdot}53$. A comparison of the three sets of curves shows that (a) $W(r, t)$ is reduced by the outward sweep of the wind, (b) the position of the maximum ($\partial W/\partial r = 0$) is moved outward, and (c) the duration of W is reduced. In Fig. 3 are plotted times at which W reaches a maximum at $r = 0$, the maximum value which W reaches at $r = 0$, and the characteristic time of the subsequent exponential asymptotic decay, as a function of wind strength Rv/κ.

The probability that a particle be observed at (r, t) is proportional to $W(r, t)$, obviously. So $W(0, t)$ is a measure of the probability that the particle be observed in the inner solar system. Hence the most probable time for a particle to be observed is the time at which

$W(0, t)$ is a maximum, plotted in Fig. 3. The time of maximum $W(0, t)$ is a measure of the time that observed cosmic ray particles have spent in the solar system prior to observation. In Fig. 4 the time in seconds to the maximum $W(0, t)$ is plotted as a function of Rv/κ for a solar wind velocity of $v = 400$ km/sec and for various values of κ in the expected range of 10^{21}–10^{22} cm²/sec. It is readily seen that for Rv/κ of the order of one, the time the average particle spends in the solar system prior to observation is of the order of days.

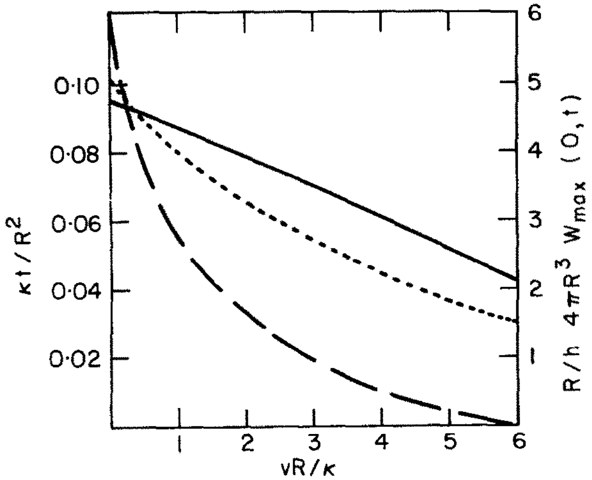

FIG. 3. THE MAXIMUM VALUE OF $W(0, t)$ IS SHOWN BY THE BROKEN LINE. THE TIME $\kappa t/R^2$ AT WHICH $W(0, t)$ REACHES ITS MAXIMUM IS SHOWN BY THE SOLID LINE. THE ASYMPTOTIC TIME IN WHICH $W(r, t)$ DECAYS BY A FACTOR OF e IS SHOWN BY THE DOTTED LINE.

Higher solar wind velocities for the same value of Rv/κ reduce the time somewhat, as may be seen from (11).

As was noted earlier the period of days which the particle spends in reaching the inner solar system is to be compared with the transit time of an hour or so in the absence of interplanetary fields. The particle remains in the solar system for days, instead of an hour, say 25 times longer because of the fields. Combining this with the fact that the cosmic ray density is lower by perhaps a factor of e in the inner solar system, leads to the conclusion that not more than about 10^{-2} of the number of particles which would pass through the inner solar system in the absence of interplanetary fields actually succeed in getting here. The other 99 per cent, or more, are excluded from the inner solar system by the outward sweep of the fields in the solar wind. We will have more to say on this later when we consider the total energy inventory of cosmic ray particles from interstellar space which run up against the outer boundary of the solar system.

(c) *Energy loss during passage*

To illustrate the energy loss of the cosmic ray particles which diffuse into the solar system consider the simple situation in which particles with energy T_0 and density N_0 fill space everywhere outside $r = R$. These particles are free to enter the region of diffusion $r < R$ and escape freely from $r = R$ following diffusion. Hence their distribution $U(r, T)$ over r and T satisfies the boundary condition

$$U(R, T) = N_0\delta(T - T_0) \tag{17}$$

The distribution in $r < R$ is determined by (4) with $\partial U/\partial t = 0$. The appropriate solution of (4) is worked out in Appendix 3. If $Rv/\kappa = 0$ it is obvious that $U(r, T) = U(R, T)$. For $Rv/\kappa \gg 1$ the asymptotic form of the energy distribution $U(0, T)$ in the inner solar system is readily obtained, and is plotted in Fig. 5 for the special case $Rv/\kappa = 5$. Figure 5 serves to illustrate the form of energy spread in the inner solar system where the particles

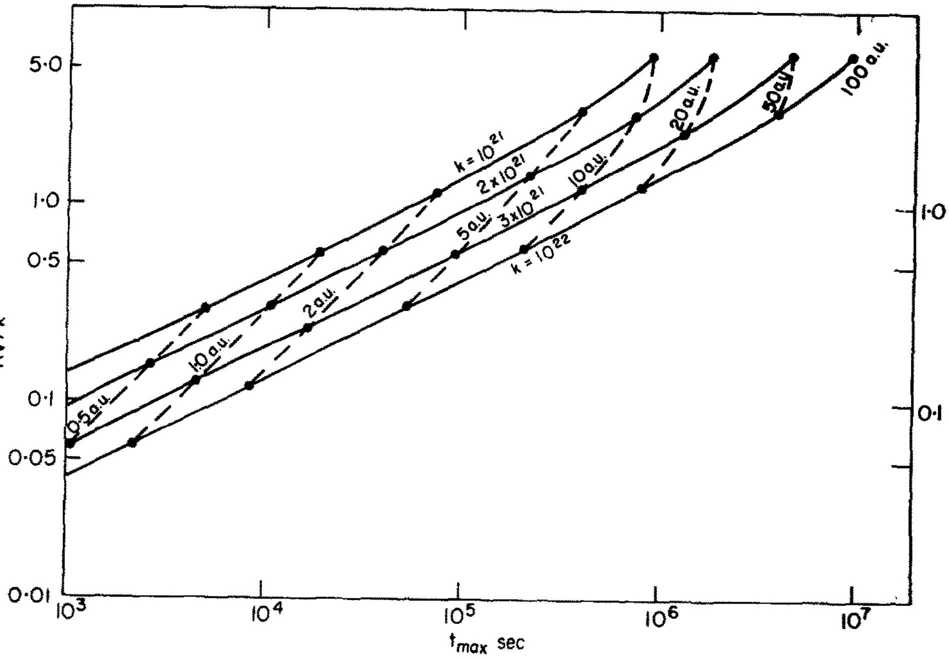

FIG. 4. THE MOST PROBABLE TIME t_{max} FOR A PARTICLE TO BE OBSERVED AT THE ORIGIN AFTER CROSSING THE OUTER BOUNDARY $r = R$ OF THE DIFFUSING REGION. THE SOLAR WIND VELOCITY IS TAKEN TO BE 400 km/sec. THE DIFFUSION COEFFICIENT κ CORRESPONDING TO EACH CURVE IS GIVEN IN cm^2/sec. THE VALUE OF R IN ASTRONOMICAL UNITS IS INDICATED ON THE BROKEN LINES. THE QUANTITY Rv/κ IS THE LOGARITHM OF THE COSMIC RAY INTENSITY N/N_0 AT THE ORIGIN.

are observed, which must be unfolded from any observed spectrum if it is desired to obtain the energy spectrum of the cosmic rays in interstellar space.

The mean energy $\langle T \rangle$ of the particles arriving at the origin is given by

$$\langle T \rangle = \frac{\int_0^{T_0} dT\, T\, U(0, T)}{\int_0^{T_0} dT\, U(0, T)}, \tag{18}$$

and is plotted in Fig. 6 as a function of Rv/κ. The breakdown of the asymptotic solution below $Rv/\kappa = 5$ is indicated by the intermittent character of the line. The dotted line is a suggested interpolation based on the asymptotic solution above $Rv/\kappa = 5$ and the point $\langle T \rangle = T_0$ at $Rv/\kappa = 0$. It is evident from Fig. 6 that for $Rv/\kappa \simeq 1$ the mean particle energy in interstellar space may be fifty per cent higher for nonrelativistic particles and twenty per cent higher for extreme relativistic particles than is observed in the inner solar system.

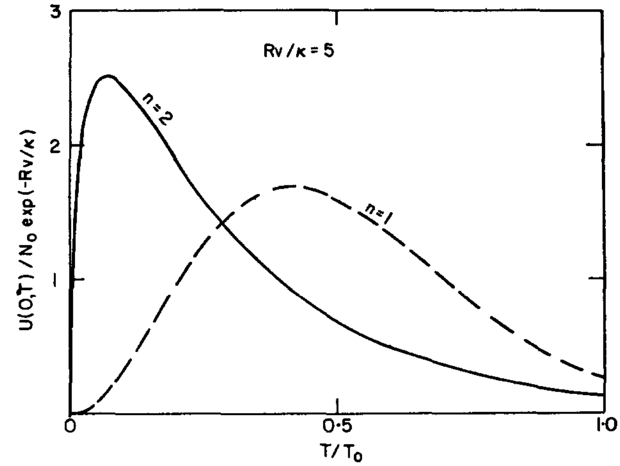

FIG. 5. THE ENERGY DISTRIBUTION $U(0, T)$ AT THE ORIGIN FOR MONOENERGETIC PARTICLES INTRODUCED STEADILY AT $r = R$ FOR THE SPECIAL CASE THAT $Rv/\kappa = 5$. THE TWO CURVES $n = 1$ AND $n = 2$ REFER TO EXTREME-RELATIVISTIC PARTICLES AND NONRELATIVISTIC PARTICLES, RESPECTIVELY.

It shows also that if Rv/κ should be significantly larger than one, the cosmic ray particles in interstellar space may be *very* much more energetic than observed here.

The reader who is interested in the energy loss to particles in interplanetary space is referred to additional illustrations elsewhere in the literature. Singer *et al.*[24] have considered the problem of deceleration without including the convective term in (4). It is possible with this omission to treat the diffusion coefficient κ as a general function of T, using the mathematical formalism of the Fermi age theory. The particle energy loss behind a blast wave from the Sun has been discussed by Parker[5]. Generally speaking, the deceleration contributes about as much as the density decrease to the observed reduction of cosmic ray intensity in the solar system.

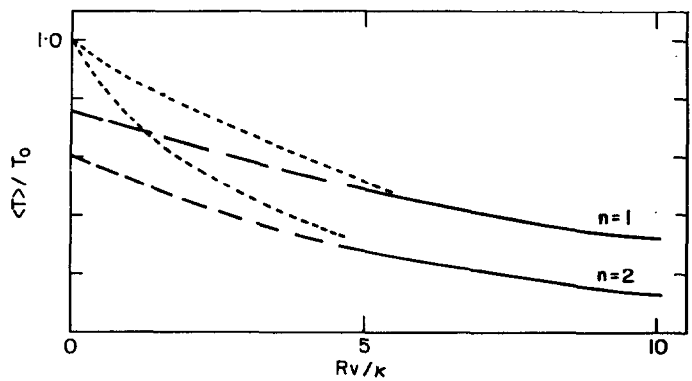

FIG. 6. THE AVERAGE PARTICLE ENERGY $\langle T \rangle$ AT THE ORIGIN FOR MONOENERGETIC PARTICLES INTRODUCED STEADILY AT $r = R$, PLOTTED AS A FUNCTION OF Rv/κ. THE SOLID CURVES ARE PLOTTED FROM THE ASYMPTOTIC FORM WHICH BREAKS DOWN SERIOUSLY BELOW $Rv/\kappa \simeq 5$, WHERE THE DOTTED LINES SERVE AS A SUITABLE INTERPOLATION. THE TWO CURVES $n = 1$ AND $n = 2$ REFER TO EXTREME-RELATIVISTIC PARTICLES AND NONRELATIVISTIC PARTICLES, RESPECTIVELY.

(*d*) *Total energy transfer between solar wind and cosmic rays*

The work P which the cosmic ray particles in $r < R$ do on the expanding fields of the solar wind is

$$P = 4\pi \int_0^R dr\, r^2 \left| \frac{dT}{dt} \right| N(r) \tag{19}$$

in unit time, where $N(r)$ is the total particle density. For stationary conditions dT/dt is given by (2) as

$$\left| \frac{dT}{dt} \right| = \frac{2n}{3} \frac{v}{r} T \tag{20}$$

and

$$N(r) = N_0 \exp\left[-(R - r)v/\kappa\right] \tag{21}$$

Now if $Rv/\kappa \lesssim 1$, then T is very roughly equal to T_0 in (20) (see Fig. 6). Only where $(R - r)v/\kappa \gg 1$ does T differ a whole lot from T_0. But in such regions $N(r)$ is extremely small, as is evident from (21), so that the error made in writing $T \cong T_0$ contributes very little to the integral in (19). Hence to a sufficient approximation

$$P \cong \frac{4\pi R^3}{3} \frac{2nN_0T_0}{R} \left(\frac{\kappa}{Rv}\right)^2 \left[1 - \left(1 + \frac{vR}{\kappa}\right) \exp\left(-\frac{Rv}{\kappa}\right)\right] \tag{22}$$

For $Rv/\kappa \ll 1$,

$$P \cong \frac{4\pi R^3}{3} \frac{nN_0T_0v}{R} \tag{23}$$

and for $Rv/\kappa \gg 1$,

$$P \cong \frac{4\pi R^3}{3} \frac{2nN_0T_0v}{R} \left(\frac{\kappa}{Rv}\right)^2 \tag{24}$$

In addition to the work P done by the cosmic ray particles on the wind, there is the work P' done by the wind on the cosmic ray particles. When a cosmic ray particle approaches the solar system and makes a collision with the magnetic fields carried in the wind, that collision is a head-on collision and the cosmic ray particle receives energy $O(T_0v/c)$ by the well known Fermi mechanism. This is only a small energy gain per particle, but so many particles are involved that $P' > P$. That is, the energy lost from each particle in the solar wind is large compared to T_0v/c, but so few particles penetrate into the solar wind that there is a net transfer of energy from the wind to the cosmic ray particles. It is shown in Appendix 6 that

$$P' = \frac{8\pi R^3}{3} \frac{N_0T_0v}{R} \tag{25}$$

for the simple nonrelativistic case. The net transfer Π is then

$$\Pi = P' - P \tag{26}$$

from the wind to the particles. For $Rv/\kappa \ll 1$, Π goes to zero, as we would expect of a slow or transparent wind. For $Rv/\kappa \gg 1$, $P \ll P'$ so that $\Pi \cong P'$. Thus Π is positive for all positive Rv/κ, indicating that in general the stellar wind regions throughout the galaxy do work on the cosmic rays. This general Fermi acceleration by stellar winds was considered some time ago by Davis[25] who showed that it probably contributes very little to the overall acceleration of cosmic rays in the galaxy. It does, however, consume a significant portion of the energy in the solar wind, which is evident from the fact that the galactic cosmic rays form a large fraction of the interstellar pressure against which the solar wind is working[4,5,26]. The kinetic energy in the solar wind is of the order of 10^{27} ergs/sec, whereas something like 10^{30} ergs/sec per star seems to be needed to maintain the galactic cosmic ray intensity.

4. PASSAGE OF ENERGETIC SOLAR PARTICLES OUT OF THE SOLAR SYSTEM

(a) *Time of passage*

Consider the escape of energetic particles from the Sun out through the interplanetary fields into interstellar space. If particles are released suddenly at the origin at time $t = 0$, the initial condition is

$$W(r, 0) = \lim_{\epsilon \to 0} \frac{\delta(r - \epsilon)}{4\pi\epsilon^2} \tag{27}$$

together with the boundary condition (16). The solution of (15) for this case may be constructed from the solution given in Appendix 1 by replacing $R - h$ there with ϵ. The decay of the particle density $W(0, t)$ at the origin is illustrated in Fig. 7 for various effective

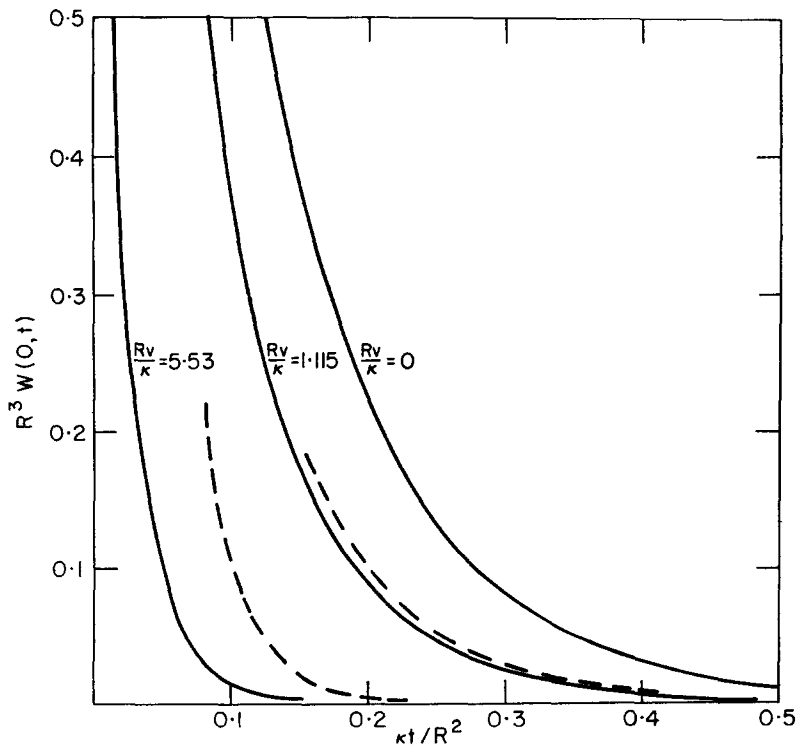

FIG. 7. THE DECAY OF THE PARTICLE DENSITY $W(0, T)$ AT THE ORIGIN IS SHOWN BY THE SOLID CURVES AS A FUNCTION OF $\kappa t/R^2$ FOLLOWING RELEASE AT THE ORIGIN AT TIME $t = 0$. THE PARAMETER Rv/κ REPRESENTS THE RELATIVE STRENGTH AND/OR EXTENT OF THE SOLAR WIND. THE BROKEN LINES REPRESENT THE DECLINE OF THE MAXIMUM PARTICLE DENSITY IN THE RADIAL DISTRIBUTION PLOTTED IN FIG. 8.

wind strengths Rv/κ. It is evident that the effect of the wind velocity v is to assist the diffusion in transporting solar particles out of the solar system. To compare the convective velocity v with the diffusion velocity, note that the particles diffuse a characteristic distance r in a time t where $r^2 = 4\kappa t$. Hence the characteristic diffusion velocity is

$$\frac{dr}{dt} = \frac{\kappa^{1/2}}{t^{1/2}} = \frac{2\kappa}{r}$$

It is evident that the diffusion velocity exceeds the convective velocity at first, and the two become equal only when $rv/\kappa = 2$. But we have pointed out that the maximum value of rv/κ, namely Rv/κ, may perhaps be only as large as 1. If this is correct, then the diffusion velocity may dominate the convective velocity at all times. Then as a first rough approximation the decay proceeds as $t^{-3/2}$ near the Sun until the particles begin to reach R in significant numbers, whereupon the decay becomes exponential as described elsewhere [5,8]. The effect of the convection (which may be large if Rv/κ should prove to be greater than one) is to hasten the onset of the steepening of $t^{-3/2}$, which leads eventually to the asymptotic exponential decline. The convection also increases the rate of the final exponential decline, as may be seen from Fig. 3.

Perhaps the most novel feature introduced by the convection is the asymptotic form of the particle distribution in space during the exponential decline. In the absence of convection this distribution is of the form sin r/r, which is a maximum at the Sun. In the presence of convection the distribution is more complicated, with a *minimum* at the Sun and a *maximum* at some distance out in space. $Rv/\kappa = 1$ puts the maximum at the orbit of Earth or beyond, as may be seen from Fig. 8. The particle density at the maximum may be considerably greater than at the origin. The particle density at the maximum as a function of time is shown in Fig. 7 by the broken lines.

In closing this discussion we remind the reader that the actual diffusion in interplanetary space is not uniformly distributed throughout the solar system, as it is in the present illustrative examples, so that one must be cautious in making detailed application to so localized a region as the space circumscribed by the orbit of Earth. Observations[5,8,27] show evidence of inhomogeneities in κ within 5–10 hr after a flare, before the solar wind has had time to sweep more than 0·1 a.u. The present calculations are intended only to illustrate the general trend caused by the wind, such as the tendency for the particle density to be a maximum at some distance out in space from the origin. For fitting data taken in the restricted volume of space open to observation near the orbit of Earth the simple models with $Rv/\kappa = 0$[5] are as good as any, because their simplicity permits other complications such as changes in κ to be considered easily too.

(*b*) *Energy loss during outward passage*

The energetic particles from the Sun are cooled adiabatically in the expanding fields carried in the solar wind, just as are the galactic cosmic ray particles. If, as an extreme case, the effective diffusion coefficient κ were so small that the particles were constrained to move with the solar wind ($Rv/\kappa \cong \infty$), then the particle energy T would vary approximately as the density to the power $n/3$, where $n = 1$ for extreme relativistic particles and $n = 2$ for nonrelativistic particles. A particle of 10^9 eV released from the flare where the gas density is say, $10^7/\text{cm}^3$, would have only about 10^5 eV at the orbit of Earth where the density is, say, $10/\text{cm}^3$. As already noted, κ appears to be sufficiently large that the particles escape more rapidly than the solar wind velocity v. To treat the energy loss in this case, suppose that particles of energy T_0 are released continuously at the origin at the rate of N/sec. The boundary condition is

$$-4\pi r^2 \kappa \frac{d}{dr} \int_0^\infty dT\, U(r, T) = N$$

or

$$U(r, T) \sim \frac{N\delta(T - T_0)}{4\pi\kappa r} \tag{28}$$

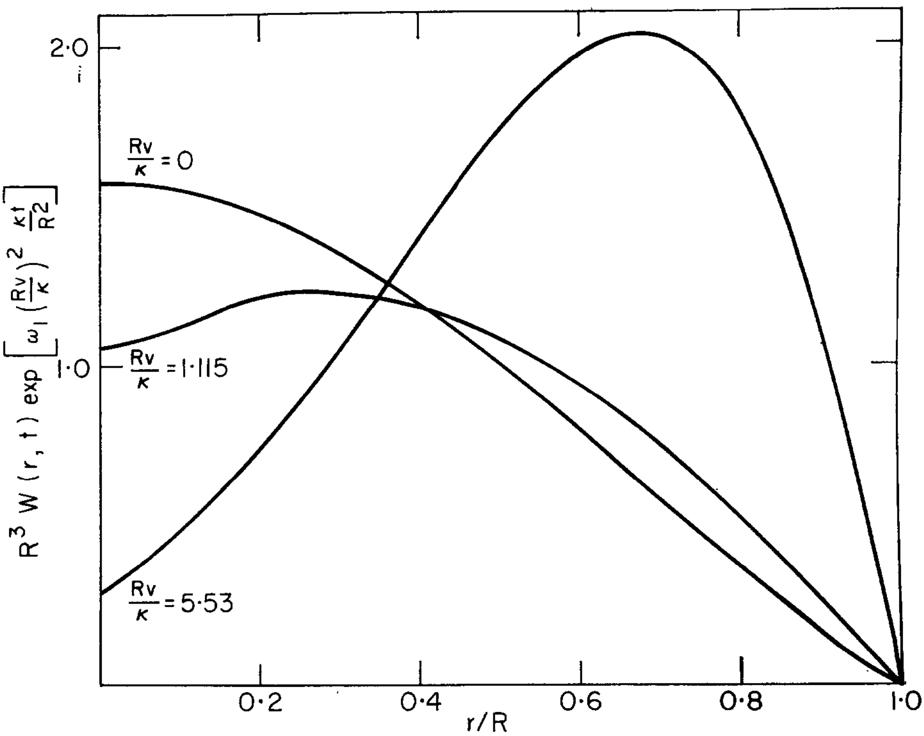

FIG. 8. THE ASYMPTOTIC RADIAL DISTRIBUTION OF THE PARTICLES FOR LARGE VALUES OF $\kappa t/R^2$ ARE PLOTTED FOR VARIOUS WIND STRENGTHS Rv/κ.

as $r \to 0$. At the outer boundary (16) prevails. Solution of (4) is given in Appendix 4. The decline of the energy T as the particles diffuse outward through the solar system is illustrated in Fig. 9 where the mean particle energy $\langle T \rangle$ is plotted as a function of radial distance r for the case that $R \gg r$. Since presumably Rv/κ is as large as $O(1)$ during the years of solar activity, it is readily seen that the energetic nonrelativistic solar particles have had their energy reduced by a factor of the order of four before they escape into interstellar space. Even extreme relativistic particles have their energy reduced by more than a factor of two. This serious deceleration must be kept in mind in theories for the origin of cosmic rays in which the cosmic rays are injected from stars and novae etc.

Since rv/κ at the orbit of Earth is presumably small, of the order of 10^{-1}, there is probably little energy reduction at Earth of particles which have diffused directly from the Sun. Figure 9 suggests only a 25 per cent reduction in nonrelativistic particle energy for $rv/\kappa = 0{\cdot}1$.

The energy lost by the energetic particles goes into increasing the velocity of the solar wind. But observations suggest that the energetic particle density probably never gets so high in interplanetary space that the wind velocity is greatly increased thereby. The

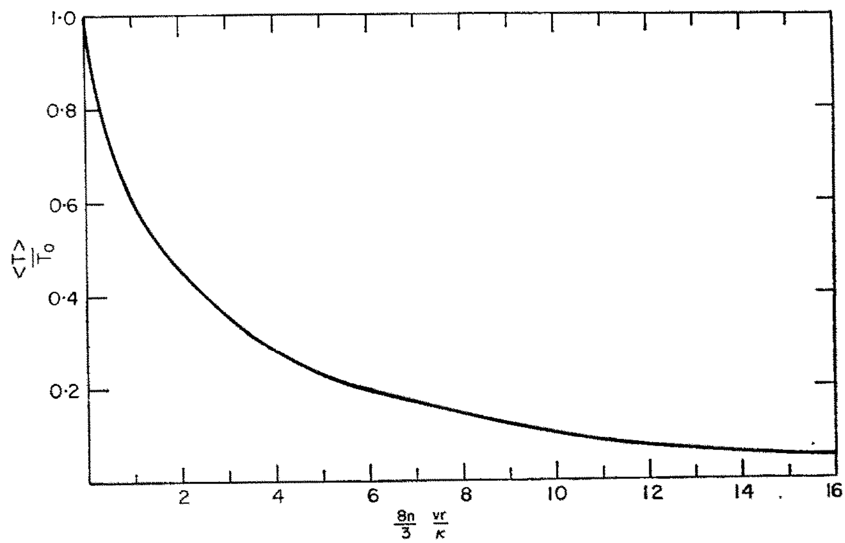

Fig. 9. The mean energy $\langle T\rangle$ of particles which have diffused from the origin, where their energy was T_0, to a radial distance r. The parameter $8n/3$ has the value $16/3$ for nonrelativistic particles and $8/3$ for extreme relativistic particles. The parameter Rv/κ is a measure of the distance to which the region of diffusion and deceleration extends.

kinetic energy density of a minimal quiet day wind of 2 hydrogen atoms/cm^3 at 400 km/sec is $0{\cdot}4 \times 10^{-8}$ ergs/cm^3, or some 4×10^3 times the normal galactic cosmic ray background.

5. ANISOTROPIC PASSAGE OF ENERGETIC PARTICLES

(*a*) *Cosmic ray modulation*

The effect of the solar wind on the passage of energetic particles in and out of the solar system has been illustrated in the preceding sections with the simplification that the diffusion coefficient κ_{ij} is isotropic, corresponding to $\nu \gg \Omega$ in (10). In this way the convection and deceleration of the particles were illustrated. In the present section we undertake to illustrate the effects of anisotropy, omitting the previous effects of convection and deceleration. Consider the situation that $\kappa_\perp \ll \kappa_\parallel$ as a consequence of the cyclotron frequency Ω being large compared to the scattering frequency ν. Then

$$\kappa_{ij} \cong \nu L^2 \frac{B_i B_j}{B^2}, \tag{29}$$

and the diffusion is limited to the direction along the lines of force of the underlying field. In this approximation the particle density along each line of force is independent of the density distribution along the neighboring lines of force. Let κ denote the diffusion coefficient along B_i. For the idealized case that the solar wind is uniform around the Sun the underlying pattern of the interplanetary field is the spiral[1,5]

$$r = \frac{v\phi}{\omega}, \quad \theta = constant \tag{30}$$

where ω is the angular velocity (3×10^{-6} radians/sec) of the Sun. It is readily shown (see Appendix 5) that a uniform cosmic ray density N_0 beyond $r = R$ with a uniform diffusion coefficient κ inside $r = R$, leads to the cosmic ray density

$$N(0, \theta) \sim N_0 \exp\left\{-\frac{vR}{\kappa}\left[1 + \frac{1}{3}\left(\frac{R\omega \sin \theta}{v}\right)^2\right]\right\} \tag{31}$$

at the origin. Noting that $r\omega/v$ is of the order of unity at the orbit of Earth, it is evident that $(R\omega \sin \theta/v)^2$ is very large compared to unity, so that the reduction for a given Rv/κ is by *much more* than the factor $\exp(-vR/\kappa)$ for isotropic diffusion. The adiabatic deceleration is also much greater for a given Rv/κ. The reason for the greater reduction is simply that the spiral path by which the particles can enter the solar system is very much longer than the radial path available when κ_{ij} is isotropic. The time to diffuse from R in to a given r is considerably increased, so that the outward convection has longer to act to reduce the particle density. It is evident that values of Rv/κ considerably less than one will account for a reduction of the cosmic ray intensity by a factor of e.

(*b*) *Escape of solar particles*

In the presence of isotropic diffusion the density of a burst of energetic particles released at the Sun dies away as $1/t^{3/2}$ [5,8] throughout the inner solar system (neglecting the convection discussed in section 3). In the present case also a burst dies away as $1/t^{3/2}$ during the initial stages, because the lines of force, along which the particles diffuse, are approximately radial near the Sun. But as the particles reach the outer regions, $(r\omega/v) \sin \theta > 1$, beyond the orbit of Earth where the field is seriously spiralled, the path length increases and the decline goes as $1/t^{3/4}$ (see Appendix 5). It is interesting to note that in a two dimensional model (cylindrical Sun etc.) the decline is $1/t$ when the field is radial and $1/t^{1/2}$ when the particles reach into the spiral. The exponent on the time appears to be reduced to one half by the spiral pattern. It is evident also that the adiabatic deceleration must be greater than computed for isotropic diffusion for a given vR/κ.

(*c*) *Effect of varying diffusion coefficient*

If it is assumed that the diffusion coefficient increases outward in proportion to r, then it is readily shown (Appendix 5) that the cosmic ray intensity has a value

$$N = N_0 \left(\frac{v}{\omega R}\right)^{v^2/\kappa\omega} \exp\left\{\frac{v^2}{2\kappa\omega}\left[1 - \left(\frac{\omega R}{v \sin \theta}\right)^2\right]\right\} \tag{32}$$

at a radial distance $r = (v/\omega) \sin \theta$ of about one a.u. where κ now denotes the diffusion coefficient at the same distance. The more general case that $\kappa \propto r^s$ is also given in Appendix 5*, where it is shown that for $s < 4$ the density of a burst of particles released at the origin declines as $1/t^{3/(4-s)}$. The decline is more rapid for larger s.

(d) Comparison of the results for isotropic and anisotropic diffusion

It is interesting to note that the constraint of the particles to the spiral lines of force has the same effect as an isotropic diffusion coefficient proportional to $(1 + u^2)^{-1}$, $u = (r\omega/v) \sin \theta$. A diffusion coefficient proportional to $(1 + u^2)\psi(u)$ makes the behavior of the particle density constrained to diffusion along the lines of force identical with the behavior in the isotropic case with the diffusion coefficient given by just $\psi(u)$. In view of these facts and the theoretical conjecture that κ must in general increase somewhat with the weakening fields at large distance from the Sun, we are unable at the present time to make any assertions whether the anisotropic diffusion along the observed lines of force at Earth extends very far beyond the orbit of Earth.

In fact the difference between the two situations is very slight so long as we are restricted to observations near the orbit of Earth: On the one hand, suppose that the particles diffuse only along the magnetic lines of force. Put $\kappa \propto u^2$ so that a burst of solar particles declines like the typical $t^{-1.5}$ (see Appendix 5, equation 17) demanded by observation. Then in order that the cosmic ray density at 1 a.u. be the fraction exp (-1) of the density at $r = R$ (as minimum requirement for the 11-year variation) it is readily shown from (Appendix 5, equation 18) that the diffusion coefficient at the orbit of Earth must have the value

$$\kappa \cong \frac{v^2}{\omega \sin \theta}\left(\frac{\omega R \sin \theta}{v} - \frac{v}{\omega R \sin \theta}\right)$$

For a 450 km/sec wind the length v/ω is just 1 a.u., so that with $\theta = \pi/2$ this yields

$$\kappa \cong 6{\cdot}75 \times 10^{20}\left(R - \frac{1}{R}\right) \text{cm}^2/\text{sec} \tag{33}$$

if R is measure in a.u. Any value of R in 5–20 a.u. gives the right order of magnitude (10^{21}–10^{22}) for κ. On the other hand, for isotropic diffusion, suppose that κ is more or less independent of radial distance from the Sun, so that flare particles again decline as $t^{-1.5}$. Then put $Rv/\kappa = 1$, so that the cosmic ray density is again the fraction exp (-1) of the interstellar density. We have

$$\kappa = 6{\cdot}75 \times 10^{20} R \text{ cm}^2/\text{sec} \tag{34}$$

for a 450 km/sec wind if R is measured in a.u. The similarity of the values of κ obtained from the two extreme situations, (33) and (34), for $R \gg 1$ is immediately evident.

The difference between the two models lies in the behavior of the interplanetary field beyond the orbit of Earth. If the spiral structure presently observed near Earth, extends far beyond Earth, then theory requires an associated rapid increase of κ with distance, say as r^2. If the spiral structure is largely obliterated by disorder, κ presumably remains more nearly uniform for a distance of at least a few a.u. It will be extremely interesting when space vehicles venture significantly beyond the orbit of Earth, to see what they show. Actually the two idealizations of a κ increasing as fast as r^2, or a κ which does not increase at all both seem rather extreme to us, so we would not be surprised to find the actual situation somewhere between the two idealized models of complete isotropy and complete anisotropy used for illustration here. Perhaps one situation prevails during one part of the cycle of solar activity, and another during the other part.

* The case that $\kappa \propto r^s$ is worked out elsewhere[5] for isotropic diffusion.

It has been pointed out elsewhere[16] that the anisotropy (scattering frequency $\ll$ cyclotron frequency) at the orbit of Earth plays an essential role in producing the diurnal effect. The diurnal effect demands a certain amount of diffusion across the magnetic lines of force beyond the orbit of Earth, but the amount is so slight that it does not appear to help resolve the present question. Theoretical models, with all the complications of convection, partial anisotropy, spatial variation of κ_{ij} etc., must await further guidance from observation.

6. SUMMARY AND CONCLUSIONS

The present study has been aimed at qualitative illustration of the physical behavior of energetic charged particles in the interplanetary magnetic fields. The random walk treatment of the particle motion has been extended to the anisotropic case of preferential diffusion along the magnetic lines of force. The recent magnetometer observations of the interplanetary magnetic fields near Earth indicate that the diffusion is in fact preferentially along the magnetic field, though we have no idea how far this extends beyond the orbit of Earth. A detailed comparison of the diffusion of energetic solar particles into interstellar space shows that preferential diffusion along the underlying spiral magnetic pattern decreases the power α of the particle density decline $1/t^{\alpha}$ after a flare to about half the value it would have if the diffusion were isotropic. For instance, the decay for a uniform diffusion coefficient κ along the spiral lines of force gives $\alpha = 3/4$, whereas isotropic diffusion gives $\alpha = 3/2$. The observed values of α following a solar flare are generally 1·5 or more. So if the diffusion coefficient were uniform, as assumed in obtaining these results, the diffusion along the spiral could be ruled out in favor of more nearly isotropic diffusion. Unfortunately the expected increase of the diffusion coefficient κ with distance from the Sun increases α, so the situation is not as clear as one might hope.

Now quite apart from the question of isotropy versus anisotropy, we were able to illustrate the inward progression of an individual cosmic ray particle from interstellar space to the orbit of Earth. The progress of individual cosmic ray particles is not something that one observes directly, but the period of time for the inward passage, which we estimate conservatively to be a few days during the years of solar activity, makes it clear that the typical cosmic ray particle loses not less than 15 per cent of its initial energy. This energy loss, and the uncertainties in it, should be taken into account in any extrapolations that are made to estimate the cosmic ray energies in interstellar space. It was also possible to show that about one in 10^2 of the galactic cosmic rays incident on the outer boundary of the solar wind succeeds in penetrating to any great depth. But that once having penetrated to the orbit of Earth, the particles remain perhaps 25 times longer in the solar system than if there were no interplanetary magnetic fields. The principal energy exchange between the wind and the galactic cosmic ray particles is the head-on collision which the reflected cosmic ray particles make with the fields in the wind. A large portion of the solar wind energy is transferred in this way into the cosmic rays which fill interstellar space.

The outward passage of energetic particles from the Sun is affected by the motion of the wind. For uniform diffusion κ the initial density decline of a burst of energetic particles is $1/t^{3/2}$. The effect of the convection by the solar wind is to increase the rate of the decay after a time, hastening the onset of the final exponential asymptotic decay, which begins when the particles reach the outer boundary $r = R$ in significant numbers. The convection reduces the characteristic time of this final decay too. The convection has the further effect of moving the maximum of the declining particle density from the Sun, where it resides in the absence of wind, outward to some position beyond the orbit of Earth during the final period of exponential decline (see Fig. 8).

The theoretical considerations taken up in this study show the kinds of questions that can be answered only by observations. First of all, it is evident that much more information of the kind illustrated in Fig. 1 needs to be accumulated near Earth. As pointed out elsewhere[15] the question of whether the magnetic irregularities are unrelated bends, or localized waves, in the lines of force, has a great deal to do with their effectiveness in scattering high energy particles. Does the field in fact follow the pattern set by these preliminary plots, with a tendency for a sharp bend every 1–10×10^6 km or so, or is the field usually smoother, or much more regular? The basic *tensor* properties of the diffusion coefficient κ_{ij} depend very much on such things. And whatever the nature of the field now, how will it, and κ_{ij}, vary over the 11- or 22-year cycle of solar activity. Then there is the question of the field, and κ_{ij} beyond the orbit of Earth. Do the irregularities in the field increase beyond Earth, or do they decrease? How does the tensor form of κ_{ij} change with radial distance from the Sun? How does the magnitude of κ_{ij} change? It will be particularly interesting when detailed studies of the time variations of solar particle intensities, which have already thrown so much light on the properties of the interplanetary field[8,10,17,18] can be carried out in association with simultaneous direct observation of the interplanetary fields.

The ultimate observational question is, of course, to determine by how much the cosmic ray density is reduced in the inner solar system below the level in interstellar space.

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REFERENCES

1. E. N. Parker, *Phys. Rev.* **109,** 1874 (1958).
2. E. N. Parker, *Astrophys. J.* **123,** 664 (1958).
3. E. N. Parker, *Astrophys. J.* **133,** 1014 (1961).
4. E. N. Parker, *Astrophys. J.* **134,** 20 (1961).
5. E. N. Parker, *Interplanetary Dynamical Processes.* Interscience, New York (1963).
6. E. N. Parker, *Phys. Rev.* **110,** 1445 (1958).
7. P. Morrison, *Phys. Rev.* **101,** 1397 (1956).
8. P. Meyer, E. N. Parker and J. A. Simpson, *Phys. Rev.* **104,** 768 (1956).
9. J. F. Steljes, H. Carmichael and K. G. McCracken, *J. Geophys. Res.* **66,** 1363 (1961).
10. K. G. McCracken, *J. Geophys. Res.* **67,** 447 (1962).
11. W. I. Axford, A. J. Dessler and B. Gottlieb, *Astrophys. J.* **137,** 1268 (1963).
12. J. A. Simpson, *Semaine D'Etude sur le Probleme du Rayonnement Cosmique dans L'Espace Interplanetaire,* p. 323–338. Pontifical Academy of Science, Vatican.
13. N. F. Ness, C. S. Scearce and J. B. Seek, presented at COSPAR, Florence, Italy (May, 1964). To be published in *J. Geophys. Res.* (1964).
14. E. J. Smith, L. Davis, P. J. Coleman and C. P. Sonett, *Science* **139,** 1099 (1962).
15. E. N. Parker, *J. Geophys. Res.* **69,** 1755 (1964).
16. E. N. Parker, *Planet. Space Sci.* **12,** 735 (1964).
17. D. A. Bryant, T. L. Cline, U. D. Desai and F. B. McDonald, *J. Geophys. Res.* **67,** 4983 (1962).
18. D. J. Hofmann and J. R. Winckler, *J. Geophys. Res.* **68,** 2067 (1963).
19. C. Y. Fan, P. Meyer and J. A. Simpson, *Phys. Rev. Letters* **5,** 272 (1960).
20. H. V. Neher and H. R. Anderson, *J. Geophys. Res.* **69,** 1911 (1964).
21. S. Chandrasekhar, *Rev. Mod. Phys.* **15,** 1 (1943).
22. E. N. Parker, *Phys. Rev.* **107,** 924 (1957).

23. M. NEUGEBAUER and C. SNYDER, *Science* **138**, 1095 (1962).
24. S. F. SINGER, H. LASTER and A. M. LENCHEK, *J. Phys. Soc. Japan* **17**, Suppl. A-II, 583 (1962).
25. L. DAVIS, *Suppl. Nuovo Cimento* **8**, 444 (1958).
26. E. N. PARKER, *Planet. Space Sci.* **9**, 461 (1962).
27. R. VOGT, *Phys. Rev.* **125**, 366 (1962).
28. G. N. WATSON, *Theory of Bessel Functions* (2nd ed.). Cambridge University Press, Cambridge (1958).
29. E. FERMI, *Phys. Rev.* **75**, 1169 (1949).
30. E. FERMI, *Astrophys. J.* **119**, 1 (1954).
31. E. N. PARKER, *Phys. Rev.* **109**, 1328 (1958).

Резюме—Дается пояснение, сопровождаемое рядом идеализированных примеров, прохода частиц космических лучей через межпланетное пространство. Формальные примеры разработаны на предположении, что энергетические частицы в межпланетном пространстве блуждают в неоднородностях в пространном межпланетном магнитном поле. Неоднородности продвигаются со скоростью приблизительной солнечному ветру. Вероятное классическое распределение может быть подведено под уравнение Фоккер-Планка. Разработано выражение общего характера для коэффициента κ_{ij} рассеяния частиц, включающее как рассеяние в магнитных неоднородностях, так и дрейфы систематического давления. Даются магнитометровые данные, полученные исследователем ХУШ, чтобы показать насколько тесно—в среднем—межпланетное магнитное поле, в спокойный день, согласуется с теоретическим спиральным углом, а также указать на то, что частицы привычно передвигаются свободнее вдоль поля, нежели поперек него, $\kappa_{\parallel}, \kappa_{\perp}$. Наблюдаемые поля обнаруживают, что коэффициент рассеяния принадлежит порядку в 10^{21}–10^{22} см^2 сек, соответствующему прежним калькуляциям в изучении космических лучей. Среднее значение в 3×10^{21} см^2 сек, заставляет предполагать, что градиент плотности космических лучей прибл. 10% на а.е. поперек орбиты Земли.

Непосредственные наблюдения межпланетного магнитного поля предоставляют возможность количественного вычисления κ_{ij}, в качестве функции энергии частиц. Первым примером, подлежащим рассмотрению, является изотропное рассеяние в сферической области $r < R$ с равномерной радиальной скоростью ветра v, и он дается в целях иллюстрации общего характера продолжительности прохода частицы космического луча через солнечную систему. Указывается, что плотность космических лучей сокращается в порядке exp $(-vR/\kappa)$ и что следовательно в годы солнечной активности vR/κ не менее чем прибл. 1 для протонов одного BeV, или около этого. Из этого следует, что галактические частицы космических лучей, ко времени наблюдения их, уже обычно провели несколько дней в солнечной системе. В течение этого периода они находятся в расширяющихся магнитных полях, несомые солнечным ветром, с адиабатически замедленной скоростью, утрачивая 15 или более процентов своей энергии к тому времени, что они подвергаются наблюдениям. Распределение энергии указано для частиц с одинаковой исходной энергией T_0 от межзвездного пространства. Наступающая вероятная волна единой частицы исчисляется в качестве функции времени, указывая как частица относится ветром.

Дается иллюстрация обратной проблемы эннргетических солнечных частиц. Солнечные частицы могут характерным образом утерять 50% своей первоначальной энергии до того, как ускользнуть в межзвездное пространство. Движение ветра, направленное наружу, перемещает их вероятное распределение наружу таким образом, что в конечном счете максимальная интенсивность солнечной частицы может находиться за пределами орбиты Земли. Направленное наружу движение ветра ускоряет понижение интенсивности солнечной частицы.

Устойчивое состояние интенсивности космических лучей вычисляется по всей сферической области $r < R$ при услории равномерности плотности N_0 космических лучей, получаемой в межзвездном пространстве. Расчет производится для изотропного κ_{ij} и он получается при условии, что магнитные неоднородности большой амплитуды и масштабом непревышающим радиус вращения частиц космических лучей, а также для анизотропного κ_{ij}, при $\kappa_{\parallel} \gg \kappa_{\perp}$, получаемого, когда поле относительно спокоино. (Наблюдения при минимуме солнечных пятен заставляют предполагать, что $\kappa_{\parallel} \gg \kappa_{\perp}$ находится у орбиты Земли.) Частицы рассеиваются лишь вдоль спиральных линий силы, когда $\kappa_{\parallel} \gg \kappa_{\perp}$ и таким образом их путь в и из солнечной системы гораздо длинее, чем в том случае, когда κ_{ij}

изотропно. В разультате, интенсивность космических лучей понижается гораздо больше для данного vR/κ_{ij}.

Относительно κ_{ij} не имеется непосредственной обсервационной информации за пределом орбиты Земли, где происходит понижение интенсивности. Однако, в распоряжении имеется косвенная информация, как тот факт, что интенсивность энергетических солнечных частиц нередко затухает, как t–g, при $g = 1{,}5 - 2.0$. Указывается, что для того, чтобы это случилось, необходимо, чтобы $|\kappa_{ij}| \propto r^s$ при $S = 0{,}0 - 0{,}5$, если κ_{ij} изотропно, продолжало быть так же анизотропно за пределом Земли, как оно наблюдалось вблизи Земли, и тогда рассеяние должно быстро увеличиваться с расстоянием от Солнца. Эти количественные характеристики могут быть легко обнаружены при наблюдениях частиц, полей и плазмы за пределом орбиты Земли.

APPENDIX 1

Particle diffusion in a radial wind

Consider the solution of the Fokker–Planck equation (1) for the probability distribution $W(r, t)$ for the special case that κ_{ij} is isotropic and subject to the boundary conditions (13) and (16). Taking v and κ to be uniform over r and t, introduce the variables $s = v^2t/\kappa$ and $\zeta = vr/\kappa$. Then

$$\frac{\partial W}{\partial s} = \frac{1}{\zeta^2}\frac{\partial}{\partial \zeta}\left[\zeta^2\left(\frac{\partial W}{\partial \zeta} - W\right)\right] \tag{1.1}$$

Put

$$W(r, t) = S(s)P(\zeta)$$

The equation is immediately separable into

$$S' + \omega S = 0 \tag{1.2}$$

and

$$P'' + \left(\frac{2}{\zeta} - 1\right)P' + \left(\omega - \frac{2}{\zeta}\right)P = 0, \tag{1.3}$$

where ω is the parameter of the separation and is taken to be real and positive. From (1.2) we have

$$S(s) = \exp(-\omega s) \tag{1.4}$$

The solution of (1.3) can be expressed in terms of confluent hypergeometric functions. It is more convenient for our present purposes, however, to consider the characteristic solutions.

Let

$$P(\zeta) = \sum_{n=0}^{\infty} A_n \zeta^{\alpha+n} \tag{1.5}$$

The indicial equation is

$$\alpha(\alpha + 1) = 0$$

We reject the case $\alpha + 1 = 0$, since $W(r, t)$ must be finite at the origin, obtaining, then,

$$P(\omega, \zeta) = C(\omega)\left[1 + \zeta + \frac{3 - \omega}{6}\zeta^2 + \frac{6 - 5\omega}{36}\zeta^3 + \frac{6\omega_2^2 - 43\omega + 30}{720}\zeta^4 + \frac{68\omega^2 - 189\omega + 90}{10800}\zeta^5 + \cdots\right] \tag{1.6}$$

for any given value of ω. Here $C(\omega)$ is an arbitrary constant to be determined by the later normalization of $P(\zeta)$. Note, for later use, that $P(0) = P'(0) = C(\omega)$. Using (1.6) to define the characteristic solutions of (1.3) we have

$$W(r, t) = \sum_{n=1}^{\infty} a_n P(\omega_n, \zeta) \exp(-\omega_n s) \tag{1.7}$$

where the ω_n are characteristic values chosen such that

$$P(\omega_n, vR/\kappa) = 0, \tag{1.8}$$

automatically satisfying the boundary condition (1.6).

Now write (1.3) for $P(\omega_a, \zeta)$ and multiply by $\zeta^2 \exp(-\zeta)P(\omega_b, \zeta)$; then write (1.3) for $P(\omega_b, \zeta)$ and multiply by $\zeta^2 \exp(-\zeta)P(\omega_a, \zeta)$; then subtract the two equations. The result may be written

$$\frac{d}{d\zeta}[\zeta^2 \exp(-\zeta)(P_aP_b' - P_bP_a')] + (\omega_b - \omega_a)\zeta^2 \exp(-\zeta)P_aP_b = 0$$

Integrate from $\zeta = 1$ to $\zeta = Rv/\kappa$ and recall (1.8). The result is

$$(\omega_b - \omega_a)\int_0^{Rv/\kappa} d\zeta\, \zeta^2 \exp(-\zeta)P_aP_b = 0, \tag{1.9}$$

which establishes the orthogonality of the characteristic solutions for $a \neq b$. Then put $s = 0$ in (1.7), multiply by $\zeta^2 \exp(-\zeta)P(\omega_m, \zeta)$, and integrate from $\zeta = 0$ to $\zeta = Rv/\kappa$. Adjust $C(\omega_m)$ such that

$$\int_0^{Rv/\kappa} d\zeta\, \zeta^2 \exp(-\zeta)P^2(\omega_m, \zeta) = 1 \tag{1.10}$$

The result is

$$a_m = \int_0^{Rv/\kappa} d\zeta\, \zeta^2 \exp(-\zeta)P(\omega_m, \zeta)W(r, 0) \tag{1.11}$$

For the initial conditions (13),

$$a_m = \frac{v^3}{4\pi\kappa^3} \exp\left[-\frac{v}{\kappa}(R - h)\right] P\left[\omega_m, \frac{v}{\kappa}(R - h)\right], \tag{1.12}$$

so that

$$W(r, t) = \frac{v^3}{4\pi\kappa^3} \exp\left[-\frac{v}{\kappa}(R - h)\right]$$

$$\times \sum_{n=1}^{\infty} P\left[\omega_n, \frac{v}{\kappa}(R - h)\right] P\left(\omega_n, \frac{vr}{\kappa}\right) \exp\left(-\frac{\omega_n v^2 t}{\kappa}\right) \tag{1.13}$$

For v^2t/κ of the order of one, or more, the first two terms in the series give an adequate approximation.

For the present problem, wherein $h \ll R$, note that

$$P\left[\omega_n, \frac{v}{\kappa}(R-h)\right] \cong P'\left[\omega_n, \frac{v}{\kappa}(R-h)\right]\frac{v(R-h)}{\kappa} + 0\left\{\left[\frac{v(R-h)}{\kappa}\right]^2\right\} \tag{1.14}$$

Numerical integration of (1.3), beginning with (1.6) at $\zeta = 0$, yields the following special cases:

(a) For $vR/\kappa = 1{\cdot}115$, $\omega_1 = 10$, $\omega_2 = 32$. The functions $P(\omega_1, \zeta)$ and $P(\omega_2, \zeta)$ are given in Table 1 with the normalization (1.10). The special values are $P'(\omega_1, Rv/\kappa) = -6{\cdot}90$, $P'(\omega_2, Rv/\kappa) = 11{\cdot}8$.

(b) For $vR/\kappa = 5{\cdot}53$, $\omega_1 = 1$, $\omega_2 = 2{\cdot}16$. The functions $P(\omega_2, \zeta)$ and $P(\omega_2, \zeta)$ are given in Table 2 with the normalization (1.10). The special values are $P'(\omega_1, \zeta) = -1{\cdot}27$, $P'(\omega_2, \zeta) = +1{\cdot}64$.

For the special case that $vR/\kappa = 0$, it is easiest to go back to equation (2), which reduces to

$$\frac{\partial W}{\partial \tau} = \frac{1}{\xi^2}\frac{\partial}{\partial \xi}\left(\xi^2 \frac{\partial W}{\partial \xi}\right) \tag{1.15}$$

with $\tau = \kappa t/R^2$, $\xi = r/R$. The general solution of this diffusion equation can be written

$$W(r, t) = \frac{1}{\xi}\sum_{n=1}^{\infty} b_n \sin n\pi\xi \exp(-n^2\pi^2\tau) \tag{1.16}$$

where

$$b_n = 2\int_0^1 d\xi\, \xi \sin n\pi\xi\, W(r, 0), \tag{1.17}$$

subject to the condition that W be finite at the origin and satisfy (16). For the initial condition (13),

$$W(r, t) = \frac{1}{2\pi R(R-h)r}\sum_{n=1}^{\infty} \sin\frac{n\pi(R-h)}{R}\sin\frac{n\pi r}{R}\exp\left(-\frac{n^2\pi^2\kappa t}{R^2}\right) \tag{1.18}$$

$$\cong \frac{h}{2R^3 r}\sum_{n=1}^{\infty}(-1)^{n-1} n \sin\frac{n\pi r}{R}\exp\left(-\frac{n^2\pi^2\kappa t}{R}\right) \tag{1.19}$$

for $h \ll R$.

The probability distribution $W(r, t)$ is plotted in Fig. 2 for the three cases $Rv/\kappa = 0$, $1{\cdot}115$, $5{\cdot}53$. The time at which $W(r, t)$ reaches a maximum at $r = 0$ is plotted in Fig. 3 in units of R^2/κ, along with the associated maximum value of $W(0, t)$, using the first two terms in the series for the approximation. The subsequent asymptotic decay times $\omega_1 R^2/\kappa$ are also given. The time for maximum $W(0, t)$ is plotted in Fig. 4 in seconds for $v = 400$ km/sec and for various values of R and κ.

Now if the region $r = R$ were in an infinite space filled with cosmic rays with number density N_0 and an isotropic distribution of their velocities, w, then under steady conditions it is readily shown from solution of (1) with $\partial W/\partial t = 0$ that the density inside $r = R$ is $N_0 \exp[-v(R-r)/\kappa]$. Keeping this fact in mind, note that the flux of isotropic cosmic ray particles inward across $r = R$ is $N_0 w/4$, so that in the time interval $(t, t + dt)$ there should be introduced $4\pi R^2 \times \frac{1}{4}N_0 w\, dt$ particles at the depth h (corresponding to about one mean free path into the diffusing region). Summing over the probability distribution of all such particles introduced prior to the time t gives the steady distribution $N_0 \exp[-v \times (R-h)/\kappa]$,

$$4\pi R^2 \times \frac{1}{4} N_0 w \int_{-\infty}^{t} d\mu W(r, t-\mu) = N_0 \exp[-v(R-h)/\kappa] \tag{1.20}$$

This equation serves to determine the correct effective value of h. For $h \ll R$ it is readily shown from (1.19) for the special case $v = 0$ that

$$\int_{-\infty}^{t} d\mu W(r, t-\mu) = \frac{h}{2\pi\kappa r} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\pi} \sin\frac{n\pi r}{R} = \frac{h}{4\pi\kappa R^2}$$

Hence, (1.20) gives

$$h = \frac{4\kappa}{Rw}$$

The usual elementary definition of κ in terms of the particle velocity w and effective mean free path λ is $\kappa = \frac{1}{3}\lambda w$, yielding $h = 4\lambda/3$ in this special case. Another example is given in the Appendix 2, where particle diffusion in a one dimensional wind is given.

TABLE 1

ζ	$Rv/\kappa = 1{\cdot}115$ $P(\omega_1, \zeta)$ $\omega_1 = 10$	$P(\omega_2, \zeta)$ $\omega_2 = 32$
0	3·08	5·92
0·10	3·33	6·21
0·20	3·52	5·86
0·25	—	5·45
0·30	3·59	4·86
0·35	—	4·14
0·40	3·52	3·31
0·5	3·34	+1·48
0·6	3·04	−0·47
0·7	2·61	−1·95
0·8	2·07	−2·43
0·85	—	−2·43
0·90	1·46	−2·31
1·00	0·79	−1·48
1·10	0·105	−0·21
1·115	0	0

TABLE 2

ζ	$Rv/\kappa = 5{\cdot}53$ $P(\omega_2, \zeta)$ $\omega_1 = 1$	$P(\omega_2, \zeta)$ $\omega_2 = 2{\cdot}16$
0	0·147	0·269
0·5	0·233	0·407
1·0	0·345	0·527
1·2	0·397	0·558
1·4	—	0·575
1·6	0·512	0·578
2·0	0·635	0·521
2·4	0·761	0·373
2·8	0·878	+0·156
3·2	0·971	−0·167
3·6	1·020	−0·483
3·8	1·025	−0·631
4·0	1·010	−0·766
4·4	0·914	−0·903
4·6	—	−0·903
4·8	0·710	−0·822
5·2	0·372	−0·497
5·53	0	0

APPENDIX 2

Particle diffusion in a one dimensional wind

The problem of the diffusion of a cosmic ray particle upstream through a wind is illustrated very simply by the one dimensional wind, in, say, the x-direction. Some of the properties of this upstream diffusion are exhibited clearly in the linear one dimensional flow whereas they are obscured by the convergence of space toward the origin in a radial flow. The Fokker–Planck equation may now be written

$$\frac{\partial W}{\partial t} + v\,\frac{\partial W}{\partial x} = \frac{\partial}{\partial x}\left(\kappa\,\frac{\partial W}{\partial x}\right) \tag{2.1}$$

Consider the solution of this equation in $x > 0$ for uniform v and κ, subject to the boundary condition that

$$W(0, t) = 0 \tag{2.2}$$

and the initial condition that

$$W(x, 0) = \delta(x - h), \tag{2.3}$$

representing a single particle entering the region $x \geq 0$ and being scattered after penetrating a distance h. Let $Q(x, p)$ represent the Laplace transform

$$Q(x, p) = \int_0^\infty dt \exp(-pt) W(x, t) \tag{2.4}$$

of the probability distribution $W(x, t)$. Then the transform of (2.1) gives

$$pQ + v\,\frac{\partial Q}{\partial x} - \kappa\,\frac{\partial^2 Q}{\partial x^2} = W(x, 0) \tag{2.5}$$

The solution is

$$Q(x, p) = A(\exp s_1 x - \exp s_2 x) + \frac{1}{\kappa(s_1 - s_2)} \int_0^x d\mu W(x, 0)[\exp s_2(x - \mu) - \exp s_1(x - \mu)] \tag{2.6}$$

where A is an arbitrary constant and the form of the solution has been chosen such that $Q(0, p)$ vanishes and (2.2) is automatically satisfied. The quantities s_1 and s_2 are

$$s_{1,2} = \frac{v}{2\kappa}\,[1 \pm (1 + 4\kappa p/v^2)^{1/2}] \tag{2.7}$$

and

$$s_1 - s_2 = \frac{v}{\kappa}\left(1 + \frac{4\kappa p}{v^2}\right)^{1/2} \tag{2.8}$$

Introducing the initial condition (2.3) leads to

$$Q(x, p) = A\,(\exp s_1 x - \exp s_2 x) + \frac{1}{\kappa(s_1 - s_2)}\,[\exp s_2(x - h) - \exp s_1(x - h)] \tag{2.9}$$

Note that $s_1 > 0$ and $s_2 < 0$. Then the constant A is determined by the requirement that $Q(x, p)$ remain finite as $x \to \infty$ and

$$Q(x, p) = \frac{\exp s_2 x}{\kappa(s_1 - s_2)}\,[\exp(-s_2 h) - \exp(-s_1 h)] \tag{2.10}$$

Inverting the transform, and skirting the branch point in $s_1 - s_2$ at $p = -v^2/4\kappa$ so that the integrand is single valued, leads ultimately to

$$W(x, t) = \frac{1}{(4\pi\kappa t)^{1/2}}\left[1 - \exp\left(-\frac{xh}{\kappa t}\right)\right]\exp\left[-\frac{(vt + x - h)^2}{4\kappa t}\right] \tag{2.11}$$

This expression $W(x, t)$ is the Green's function for (2.1) subject to the boundary condition (2.2). From it can be generated the probability distribution for the introduction of particles in any general pattern in space and time.

The probability Π that the particle is still in the region of diffusion after a time t is

$$\Pi(t) = \int_0^\infty dx\, W(x, t) = \frac{1}{2}\left\{1 - \operatorname{erf}\left[\frac{vt - h}{(4\kappa t)^{1/2}}\right] - \exp\frac{hv}{\kappa}\left[1 - \operatorname{erf}\left[\frac{vt + h}{(4\kappa t)^{1/2}}\right]\right]\right\} \tag{2.12}$$

For small t ($4\kappa t \ll h^2$),

$$\Pi(t) \cong 1 - \left(\frac{\kappa t}{\pi h^2}\right)^{1/2}\exp\left(-\frac{h^2}{4kt}\right)\left(1 + \exp\frac{hv}{\kappa}\right) \tag{2.13}$$

For large t

$$\Pi(t) \sim \left(\frac{\kappa}{\pi v^2 t}\right)^{1/2}\left(1 - \exp\frac{hv}{\kappa}\right)\exp\left(-\frac{v^2 t}{4\kappa}\right) \tag{2.14}$$

If particles are introduced steadily at $x = h$ at a rate $N_0 w/4$, as would be the case for an isotropic distribution with density N_0 outside ($x < 0$), the distribution after a long time approaches the steady value

$$\pi(x) = \frac{1}{4}N_0 w\int_{-\infty}^{t} d\mu\, W(x, t - \mu) = \frac{1}{4}N_0\frac{w}{v}\left(\exp\frac{vh}{\kappa} - 1\right)\exp\left(-\frac{vx}{\kappa}\right), \tag{2.15}$$

which is, of course, precisely the stationary solution of (2.1). In order that $\pi(0) = N_0$, as it would for an isotropic particle distribution in $x < 0$, we must put

$$\exp\frac{vh}{\kappa} - 1 = \frac{4v}{w},$$

or since $vh/\kappa \ll 1$,

$$h = \frac{4\kappa}{w}$$

$$= \frac{4}{3}\lambda, \tag{2.16}$$

which is the same result obtained in Appendix 1 for the spherical case.

It is instructive to take a brief look at the form of the probability distribution $W(x, t)$. The distribution is essentially a wave, which starts as a very sharp spike at $x = h$ when

$t = 0$, and progresses upstream from there, broadening and dying out as it goes. Consider the situation that h is small compared to the distance x, appropriate to the problem of the diffusion of cosmic rays into the solar wind. Then define the dimensionless space and time coordinates

$$\eta = \frac{x}{h}, \tau = \frac{\kappa t}{h^2}, \tag{2.17}$$

and the parameter

$$\alpha = \frac{vh}{\kappa} \tag{2.18}$$

We are interested in values of x such that vx/κ is not less than $O(1)$. Hence, since $\eta \gg 1$ and $vx/\kappa = \alpha\eta$, it follows that $\alpha \ll 1$. In terms of η, μ and α (2.11) can be written

$$W(x, t) = \frac{1}{(4\pi h^2\tau)^{1/2}}\left[1 - \exp\left(-\frac{\eta}{\tau}\right)\right]\exp\left[-\frac{(\alpha\tau + \eta - 1)^2}{4\tau}\right]$$

The maximum of this function, where $\partial W/\partial x = 0$, lies at the point where

$$\frac{\alpha\tau + \eta - 1}{\alpha\tau + \eta + 1} = \exp\left(-\frac{\eta}{\kappa}\right)$$

Since $\alpha\tau$ and η are both large compared to one, it follows at once that η/τ must be very small compared to one, permitting expansion of the exponential in a power series. Keeping terms first order in η/τ in the expansion yields the quadratic

$$\eta^2 + \alpha\tau\eta - 2\tau = 0$$

from which it follows that the maximum has the position

$$\eta = \frac{\alpha\tau}{2}\left\{\left[1 + \frac{8}{\alpha^2\tau}\right]^{1/2} - 1\right\}$$

For intermediate times when, $\tau^2 = 1/\alpha^2$, the maximum is at $\eta = 1/\alpha$. For late times, $\tau^2 \gg 1/\alpha^2$, the maximum approaches the limiting position $\eta = 2/\alpha$, in a manner given by the asymptotic relation

$$\eta \sim \frac{2}{\alpha}\left[1 - \frac{2}{\alpha^2\tau} + O\left(\frac{1}{\alpha^3\tau^2}\right)\right]$$

For such values of time (2.11) may be approximated as

$$W(x, t) \sim \frac{1}{(4\pi h^2)^{1/2}}\left[\eta\exp\left(-\frac{\alpha\eta}{2}\right)\right]\left[\frac{1}{\tau^{3/2}}\exp\left(-\frac{\alpha^2\tau}{4}\right)\right],$$

so that the profile η exp $(-\alpha\eta/2)$ is stationary in space and decays away essentially exponentially with time. The crest of the probability wave is held at $x = 2\kappa/v$ by the sweep of the wind. Only the exponential tail of the wave extends to larger values of x.

APPENDIX 3

The energy loss of a cosmic ray particle diffusing in through the solar wind

Consider the solution of the Fokker–Planck equation (4) for the probability distribution $U(r, t, T)$ subject to the boundary condition (16) and the initial condition

$$U(r, O, T) = \frac{\delta[r-(R-h)]\delta(T-T_0)}{4\pi(R-h)^2} \tag{3.1}$$

representing a spherical shell at $r = R - h$ containing one particle with an energy T_0;

$$4\pi \int_0^R dr\, r^2 \int_0^\infty dT\, U(r, O, T) = 1. \tag{3.2}$$

In terms of $s = v^2t/\kappa$ and $\zeta = vr/\kappa$ (4) may be written

$$\frac{\partial U}{\partial s} = \frac{2n}{3}\frac{1}{\zeta}\frac{\partial}{\partial T}(TU) + \frac{1}{\zeta^2}\frac{\partial}{\partial \zeta}\left[\zeta^2\left(\frac{\partial U}{\partial \zeta} - U\right)\right] \tag{3.3}$$

upon ignoring the energy dependence of n and κ.

The time dependent problem may be treated using the Mellin transform over the energy T,

$$M(r, t, q) = \int_0^\infty dT\, T^{q-1}\, U(r, t, T) \tag{3.4}$$

Then assuming that $U(r, T, t)T^q$ vanishes as $T \to \infty$ for all $q > 0$, the Mellin transform of (3.3) is

$$\frac{\partial M}{\partial s} = -\frac{2n(q-1)}{3\zeta} M + \frac{1}{\zeta^2}\frac{\partial}{\partial \zeta}\left[\zeta^2\left(\frac{\partial M}{\partial \zeta} - M\right)\right] \tag{3.5}$$

in which we put

$$M(r, q, t) = G(\zeta)\exp(-\omega s) \tag{3.6}$$

where

$$0 = \frac{d^2G}{d\zeta^2} + \left(\frac{2}{\zeta} - 1\right)\frac{dG}{d\zeta} + \left\{\omega - \frac{2}{\zeta}\left[1 + \frac{n(q-1)}{3}\right]\right\} G, \tag{3.7}$$

which is a differential equation of the same form as considered in Appendix 1. Its solution is readily effected by the methods given there.

Fortunately the stationary solution of (3.3) is sufficient for the present purposes, with the boundary condition (17), which states merely that all the particles are introduced with an energy T_0 at the boundary $r = R$ and escape freely from that boundary thereafter. It is readily shown, if

$$U(r, T) = f(T)R(\zeta) \tag{3.8}$$

and*

$$\frac{1}{f}\frac{d}{dT}(Tf) = -i\alpha, \tag{3.9}$$

* It is easily shown, by replacing $i\alpha$ with a in (3.9), that any spectrum of the form T^{-a} fed in at $r = R$ is preserved throughout the region.

that

$$\frac{d^2R}{d\zeta^2} + \left(\frac{2}{\zeta} - 1\right)\frac{dR}{d\zeta} - 2\left(1 + \frac{in\alpha}{3}\right)\frac{R}{\zeta} = 0 \tag{3.10}$$

Hence, if $U(r, T)$ is to be finite at the origin, the solution is of the form

$$U(r, T) = \frac{T_0}{T}\int_{-\infty}^{+\infty} d\alpha\, g(\alpha) \exp\left(-i\alpha \ln \frac{T}{T_0}\right) {}_1F_1\left[2\left(1 + \frac{in\alpha}{3}\right); 2\ ;\ \zeta\right], \tag{3.11}$$

where ${}_1F_1$ represents the confluent hypergeometric function. Inverting the Fourier transform and using (9) yields

$$U(r, T) = \frac{N_0}{2\pi T}\int_{-\infty}^{+\infty} d\alpha \exp\left(-i\alpha \ln \frac{T}{T_0}\right) \frac{{}_1F_1[2(1 + in\alpha/3);\ 2;\ \zeta]}{{}_1F_1[2(1 + in\alpha/3);\ 2;\ \zeta]} \tag{3.12}$$

To determine $U(r, T)$, we must evaluate the integral on the right hand side of (3.12). It is readily shown that (3.12) reduces to

$$U(r, T) = \frac{N_0}{T}\,\delta\left(\ln \frac{T}{T_0}\right) = N_0\delta(T - T_0) \tag{3.13}$$

in the limit as $Rv/\kappa \to 0$. This demonstrates the obvious fact that there is no deceleration in the limit as Rv/κ vanishes. At the opposite extreme, $Rv/\kappa \gg 1$, use the asymptotic expansion for the confluent hypergeometric function, obtaining

$$U(r, T) = \frac{N_0}{2\pi T}\exp\left(-\frac{Rv}{\kappa}\right)\int_{-\infty}^{+\infty} d\alpha \exp(-i\alpha S)\Gamma(2 + i2n\alpha/3)$$
$$\times\ {}_1F_1[2(1 + in\alpha/3);\ 2;\ \zeta]\left[1 + \frac{2in\alpha}{3}\left(1 + \frac{2in\alpha}{3}\right)\frac{\kappa}{Rv} + O(\kappa^2/R^2v^2)\right] \tag{3.14}$$

where

$$S = \ln\left[\frac{T}{T_0}\left(\frac{Rv}{\kappa}\right)^{2n/3}\right] \tag{3.15}$$

It will be sufficient to evaluate $U(r, T)$ in the neighborhood of the origin, since present observations are confined to the inner solar system. The first term in the asymptotic expansion is readily integrated. Write the gamma function as an Eulerian integral of the second kind, and reverse the order of integration. Then

$$U(O, T) \sim \frac{N_0}{2\pi T}\exp\left(-\frac{Rv}{\kappa}\right)\int_0^{\infty} d\mu\, \mu \exp(-\mu)$$
$$\times \int_{-\infty}^{+\infty} d\alpha \exp\left[-i\alpha\left(S - \frac{2n}{3}\ln \mu\right)\right]$$
$$\sim \frac{N_0}{T}\exp\left(-\frac{Rv}{\kappa}\right)\int_{\infty}^{\infty} du\, u \exp(-u)\, \delta\left(S - \frac{2n}{3}\ln u\right)$$
$$\sim \frac{3N_0}{2nT_0}\left(\frac{Rv}{\kappa}\right)^2\left(\frac{T}{T_0}\right)^{3/n-1}\exp\left\{-\frac{Rv}{\kappa}\left[1 + \left(\frac{T}{T_0}\right)^{3/2n}\right]\right\} \tag{3.16}$$

where now $Rv/\kappa \gg 1$. The error involved in this asymptotic form may be seen from the fact that the asymptotic $U(0, T)$ does not vanish identically at energies above T_0. The magnitude of the error is demonstrated by

$$U(0, T_0) \sim \frac{3N_0}{2nT_0}\left(\frac{Rv}{\kappa}\right)^2 \exp\left(-\frac{2Rv}{\kappa}\right)$$

which is small for $Rv/\kappa \gg 1$, of course. The asymptotic energy spectrum (3.16) is plotted in Fig. 5 for the special case that $Rv/\kappa = 5$ by way of illustration.

The mean particle energy at the origin is defined by (18). Using (3.16) it is readily shown that

$$\langle T\rangle = T_0 \frac{\gamma(2 + 2n/3, 1)}{(Rv/\kappa)^{2n/3}[1 - (1 + Rv/\kappa)\exp(-Rv/\kappa)]} \tag{3.17}$$

where $\gamma(\alpha, x)$ represents the incomplete gamma function, $\int_0^x dt\, t^{\alpha-1}\exp(-t)$. The mean energy $\langle T\rangle$ is plotted in Fig. 6 as a function of Rv/κ using the asymptotic form (3.17) for large Rv/κ. The interpolation for intermediate Rv/κ is based on (3.17), and $\langle T\rangle = T_0$ at $Rv/\kappa = 0$.

APPENDIX 4

The energy loss of an energetic solar particle diffusing out through the solar wind

Consider the solution of the Fokker–Planck equation (3.3) under stationary conditions, neglecting convective transport (which is small if $Rv/\kappa < 2$), for the probability distribution $U(r, T)$ subject to the boundary condition (16) and (28). Separating the variables as in Appendix 3 it is readily shown that the general solution, subject only to the condition (16) is

$$\begin{aligned} U(r, T) = \frac{T_0}{T\zeta^{1/2}}\int_{-\infty}^{+\infty} d\alpha\; C(\alpha)\exp\left(-i\alpha\ln\frac{T}{T_0}\right) \\ \times \{[ker_1(p\zeta^{1/2}) + i\,kei_1(p\zeta^{1/2})][ber_1(p\zeta_0^{1/2}) + i\,bei_1(p\zeta_0^{1/2})] \\ -[ber_1(p\zeta^{1/2}) + i\,bei(p\zeta^{1/2})][ker_1(p\zeta_0^{1/2}) + i\,kei_1(p\zeta_0^{1/2})]\} \end{aligned} \tag{4.1}$$

where $\zeta = rv/\kappa$, $\zeta_0 = Rv/\kappa$, and $p = (8n\alpha/3)^{1/2}$. The function $C(\alpha)$ is arbitrary, and is to be fixed by the boundary conditions. Introducing (28) and inverting the Fourier transform yields

$$C(\alpha) = -\left(\frac{8n}{3}\right)^{1/2}\frac{Nv\alpha^{1/2}\exp(-i\pi/4)}{8\pi^2\kappa^2T_0[ber_1(p\zeta^{1/2}) + i\,bei_1(p\zeta^{1/2})]} \tag{4.2}$$

This form for $C(\alpha)$ in (4.1) then yields the particle distribution over space and energy.

It is sufficient for the present purposes to consider the particle distribution $W(r)$ over space

$$W(r) = \int_0^\infty dT\, U(r, T) \tag{4.3}$$

and the mean particle energy $\langle T\rangle$ given by

$$W(r)\langle T\rangle = \int_0^\infty dT\, T\, U(r, T) \tag{4.4}$$

Integrating (4.1) over T from $T = 0$ to $T = \infty$ leads to the integral

$$\int_{-\infty}^{+\infty} d\left(\ln \frac{T}{T_0}\right) \exp\left(-i\alpha \ln \frac{T}{T_0}\right) = 2\pi\delta(\alpha)$$

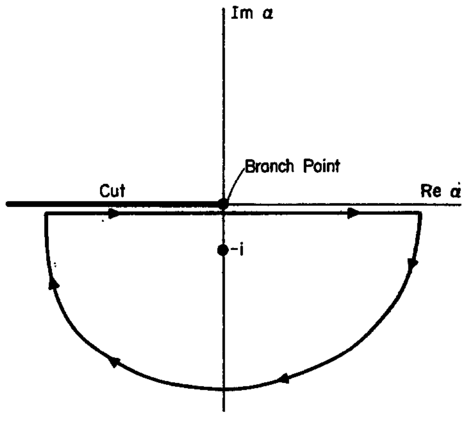

FIG. 10. A SKETCH OF THE CUT α-PLANE AND THE CONTOUR AROUND WHICH THE INTEGRATION OF (5.1) EXTENDS IN THE CONSTRUCTION OF (5.6).

The integration over α then gives

$$W(r) = \frac{N}{4\pi\kappa}\left(\frac{1}{r} - \frac{1}{R}\right) \tag{4.5}$$

This same result is readily obtained by integrating (3.3) over T, yielding

$$\frac{\kappa}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial W}{\partial r}\right) = 0,$$

which, when solved subject to the condition that there are N particles per second introduced at the origin, yields (4.5).

To compute $\langle T\rangle$ multiply (4.1) by T and integrate from $T = 0$ to $T = T_0$*. Note that

$$\int_0^{T_0} dT \exp\left(-i\alpha \ln \frac{T}{T_0}\right) = \frac{iT_0}{i + \alpha}$$

In order to carry out the integration over α it is necessary to note the branch point in the integrand of (4.1) at $\alpha = 1$, suggesting that the complex α-plane should be cut along the negative real axis. In order that the integral converge as $\alpha \to -\infty$, it is necessary to integrate along a contour lying below the cut, as indicated in Fig. 10. This is easily demonstrated by putting $\alpha = s \exp(-i\pi)$ in the asymptotic forms of the Bessel functions of

* There are no particles with energy higher than T_0.

large argument. In fact it is readily shown that the integrand goes to zero exponentially as infinity is approached in any direction below the real axis. It follows that if we should close the contour with a semicircle of infinite radius below the real axis, the integration around the semicircle gives zero. It follows from Cauchy's theorem that the integral is given by the residue of the pole at $\alpha = -i$ enclosed by the contour. The result is

$$\int_0^{T_0} dT\, T\, U(r, T) = \frac{NvT_0}{4\pi\kappa^2}\frac{8n}{3}\frac{1}{\rho^{1/2}}\left\{K_1(\rho^{1/2}) - K_1(\rho^{1/2})\frac{I_1(\rho^{1/2})}{I_1(\rho^{1/2})}\right\} \tag{4.6}$$

in terms of the radial variable $\rho = (8n/3)\zeta$. The average particle energy is

$$\langle T\rangle = T_0\frac{\rho^{1/2}}{(1-\rho/\rho_0)}[K_1(\rho^{1/2}) - K_1(\rho_0{}^{1/2})I_1(\rho^{1/2})/I_1(\rho_0{}^{1/2})] \tag{4.7}$$

In the limit as $R \to \infty$, this reduces to

$$\langle T\rangle = T_0\rho^{1/2}K_1(\rho^{1/2}) \tag{4.8}$$

at all finite r.

APPENDIX 5

Anisotropic diffusion

When the scattering frequency ν is small compared to Ω, the particle diffusion is limited to the one dimensional space along the magnetic lines of force. The diffusion coefficient is given by (29). For the idealized case that the wind is uniform around the Sun the field density $B(r, \theta, \phi)$ at a point (r, θ, ϕ) out in interplanetary space is related to the field $B(a, \theta, \phi^*)$ at $r = a$ near the Sun by

$$\phi^* = \phi + r\omega/v, \tag{5.1}$$

$$B(r, \theta, \phi) = B(a, \theta, \phi^*)\left(\frac{a}{r}\right)^2\left[1 + \left(\frac{r\omega\sin\theta}{v}\right)^{1/2}\right] \tag{5.2}$$

The line of force through (r, θ, ϕ) is given by $\phi^* =$ constant in (5.1). Arc length along the line of force is

$$ds = dr\left[1 + \left(\frac{r\omega\sin\theta}{v}\right)^2\right]^{1/2} \tag{5.3}$$

The relative cross sectional area A of a tube of flux is inversely proportional to B. The particle flux along a tube of flux is $-A\kappa\partial N/\partial s$ so that the accumulation at any given point is

$$A\frac{\partial N}{\partial t} = \frac{\partial}{\partial s}\left(A\kappa\frac{\partial N}{\partial s}\right)$$

There is in addition the accumulation from the divergence of the convective flux vN, so that altogether

$$\frac{\partial N}{\partial t} = \frac{1}{A}\frac{\partial}{\partial s}\left(A\kappa\frac{\partial N}{\partial s}\right) - \frac{1}{r^2}\frac{\partial}{\partial r}(r^2vN)$$

For a uniform diffusion coefficient along the tube of flux ($\partial\kappa/\partial s = 0$) this reduces to

$$\frac{\partial N}{\partial \tau} = \frac{1}{u^2}\left[\frac{\partial}{\partial u}\left(\frac{u^2}{1+u^2}\frac{\partial N}{\partial u}\right) - \frac{v^2}{\kappa\omega \sin\theta}\frac{\partial}{\partial u}(u^2 N)\right] \tag{5.4}$$

where

$$\tau = t\kappa\left(\frac{\omega \sin\theta}{v}\right)^2 \tag{5.5}$$

$$u = \frac{r\omega \sin\theta}{v} \tag{5.6}$$

Thus u is distance measured in units of the distance ($v/\omega \cong 1$ a.u.) out to where the spiral makes an angle of 45° with the radial direction, and τ is time measure in units of the time to diffuse the unit of distance v/ω.

For stationary conditions ($\partial N/\partial \tau = 0$) integration of (5.4) yields

$$N(u, \theta) = N_0 \exp\left\{-\frac{v}{\kappa}(R-r)\left[1 + \frac{1}{3}\left(\frac{\omega \sin\theta}{v}\right)^2 (R^2 + rR + r^2)\right]\right\} \tag{5.7}$$

for the boundary condition that the cosmic ray density has the uniform value N_0 at $r = R$ and there are no sources or sinks in $r < R$.

Consider the release of a burst of n particles/steradian from the Sun at time $t = 0$. Up to a time $\tau \cong 1$, the particles are close to the Sun ($u < 1$) and the Fokker–Planck equation (5.4), reduces to (1.1) whose solution was discussed in Appendix 1 and section 3. Anisotropy plays no role in the radial diffusion of the particles in this initial period because the field, along which they are diffusing, is radial*. Only when u becomes greater than one ($u \cong 1$ at about the orbit at Earth) and the spiral field significantly oblique does the anisotropy begin to make a difference in the radial progress of the particles. This occurs when τ becomes greater than one. For $\tau \gg 1$ the oblique path at large u impedes the outward diffusion so much that the particle density at smaller u becomes nearly uniform. Hence the particle density at small u becomes independent of the form of the diffusion term on the right hand side of (5.4) at small u. It follows that the particle density is given approximately at all u by the limiting form of (5.4) for large u

$$\frac{\partial N}{\partial \tau} \cong \frac{1}{u^2}\left[\frac{\partial^2 N}{\partial u^2} - \frac{v^2}{\kappa\omega \sin\theta}\frac{\partial}{\partial u}(u^2 N)\right] \tag{5.8}$$

for $\tau \gg 1$.

The effect of convection has been discussed in previous sections, so there is no reason to include it again here to complicate the effects of the anisotropy. In the limit of small wind velocity the convective term, with the coefficient $v^2/\kappa\omega \sin\theta$, drops out of the right hand side of (5.8) and the equation simplifies to †

$$\frac{\partial N}{\partial \tau} \cong \frac{1}{u^2}\frac{\partial^2 N}{\partial u^2} \tag{5.9}$$

* The field prevents the particles from spreading laterally around the Sun, of course.

† It is readily shown that in order of magnitude $v^2/\kappa\omega = (v/w)(v/\omega L)$. For the actual wind $L \cong 0{\cdot}1 v/\omega$ and $v/w = O(10^{-3})$, so that $v^2/\kappa\omega = O(10^{-2})$. The form (5.9) is valid until a significant portion of the particles reach a distance u given by $(\kappa\omega/v^2)^{1/3}$ which is of the order of 3 a.u. for $\kappa = 10^{22}$ cm²/sec and $v = 300$ km/sec. Thus conditions in the wind inside the orbit of Earth often approximate to the limiting case of $\kappa\omega/v^2 \rightarrow \infty$.

It is easy to show that the time independent solution of this equation can be written as

$$N(u, \tau) = u^{1/2} \int_0^\infty d\sigma\, \sigma[f(\sigma)J_{1/4}(\sigma u^2) + g(\sigma)J_{-1/4}(\sigma u^2)] \exp(-4\sigma^2\tau) \tag{5.10}$$

along any given line of force. The functions $f(\sigma)$ and $g(\sigma)$ are to be determined by the boundary conditions, which are that $N(\infty, \tau) = 0$ and

$$N(u, O) = \lim_{\epsilon \to \infty} n \frac{\delta(r - \epsilon)}{\epsilon^2} \tag{5.11}$$

Since $J_{1/4}(\sigma u^2)$ vanishes at the origin as $u^{1/2}$, there is no contribution to this mode by the initial particle injection at $u = 0$. The boundary condition $N(\infty, \tau) = 0$ prohibits the existence of the $J_{1/4}$ modes, as is readily shown by putting in a nonvanishing $f(\sigma)$ and carrying out the indicated integration over γ. So put $f(\sigma) = 0$. Inverting the Fourier–Bessel transform then gives

$$g(\sigma) = \frac{2^{5/4}n}{\Gamma(3/4)} \left(\frac{\omega \sin\theta}{v}\right)^3 \frac{1}{\sigma^{1/4}} \tag{5.12}$$

so that

$$\begin{aligned} N(u, \tau) &= \frac{2^{5/4}n}{\Gamma(3/4)} \left(\frac{\omega \sin\theta}{v}\right)^3 u^{1/2} \int_0^\infty d\sigma\, \sigma^{3/4} J_{-1/4}(\sigma u^2) \exp(-4\sigma^2\tau) \\ &= \frac{n}{2\Gamma(3/4)} \left(\frac{\omega \sin\theta}{v}\right)^3 \frac{1}{\tau^{3/4}} \exp\left(-\frac{u^4}{16\tau}\right) \end{aligned} \tag{5.13}$$

(see Watson[28]). Note the flat distribution across $u < 1$, and the $t^{-3/4}$ decline of the particle density at small u compared with the $t^{-3/2}$ for isotropic diffusion.

It is interesting to work out the same problem in two dimensions, for a cylindrical Sun revolving about its axis. The lines of force along which the diffusion takes place still have the form $\tilde{\omega} = v\phi/\omega$ where $\tilde{\omega}$ represents radial distance from the axis, but they do not open up so rapidly because the field density is now

$$B(\tilde{\omega}, \phi) = B(a, \phi^*) \frac{a}{\tilde{\omega}} \left[1 + \left(\frac{\tilde{\omega}\omega}{v}\right)^2\right]^{1/2}$$

and the Fokker–Planck equation is

$$\frac{\partial N}{\partial q} = \frac{1}{\xi} \frac{\partial}{\partial \xi} \left(\frac{\xi}{1 + \xi^2} \frac{\partial N}{\partial \xi}\right) \tag{5.14}$$

in place of (5.4), neglecting convection. Here $q = \kappa\Omega^2 t/v^2$ and $\xi = \tilde{\omega}\omega/v$, having the same significance as τ and u in the previous problem. For $\xi \gg 1$,

$$\frac{\partial N}{\partial q} \cong \frac{1}{\xi} \frac{\partial}{\partial \xi} \left(\frac{1}{\xi} \frac{\partial N}{\partial \xi}\right),$$

whose solution is readily shown to be

$$N(\xi, q) = \int_0^\infty d\sigma[F(\sigma) \sin \sigma\xi^2 + G(\sigma) \cos \sigma\xi^2] \exp(-4\sigma^2 q)$$

With n particles per radian injected at the origin at time $t = 0$, it is readily shown that $N(\xi, O) = (n/\epsilon)\delta(\tilde{\omega} - \epsilon)$ and

$$G(\sigma) = \frac{4n\omega^2}{\pi v^2}$$

and

$$N(\xi, q) = \frac{n}{\pi^{1/2}} \left(\frac{\omega}{v}\right)^2 \frac{1}{q^{1/2}} \exp\left(-\frac{\xi^4}{16q}\right) \tag{5.15}$$

The density declines as $t^{-1/2}$, compared with t^{-1} for isotropic diffusion.

Finally consider the diffusion along the spiral field in a spherical wind again, with the additional complication that the diffusion coefficient along the magnetic lines of force has the value κ_1 at $u = 1$ and $\kappa_1 u^s$ elsewhere, where s is a pure number. Then defining τ as $(t\kappa_1\omega^2/v^2) \sin^2 \theta$, the Fokker–Planck equation becomes

$$\frac{\partial N}{\partial \tau} = \frac{1}{u^2} \frac{\partial}{\partial u} \left(u^s \frac{\partial N}{\partial u}\right) \tag{5.16}$$

in place of (5.9). The solution for $s < 4$ which is nonvanishing, but finite, at the origin is

$$N(u, \tau) = u^{(1-s)/2} \int_0^\infty d\sigma\, \sigma\, h(\sigma)\, J_{(s-1)/(4-s)}\left(\frac{2\sigma}{4-s} u^{(4-s)/2}\right) \exp(-\sigma^2\tau)$$

For the initial condition (5.11) it is readily shown that

$$h(\sigma) = \frac{2n}{(4-s)^{3/(4-s)}} \left(\frac{\omega \sin \theta}{v}\right)^3 \frac{1}{\Gamma[3/(4-s)]\sigma^{(1-s)/(4-s)}}$$

Performing the indicated integration, it is readily shown that

$$N(\zeta, \tau) = \frac{n}{(4-s)^{(2+s)/(4-s)}\Gamma[3/(4-s)]} \left(\frac{\omega \sin \theta}{v}\right)^3 \frac{1}{\tau^{3/(4-s)}} \exp\left[-\frac{u^{4-s}}{(4-s)^2\tau}\right] \tag{5.17}$$

The density decays as $t^{-3/(4-s)}$. The decline is more rapid when the diffusion coefficient increases with radial distance ($s > 0$). This effect has been discussed elsewhere[5] for isotropic diffusion. The stationary solution of (5.4) for this same case $\kappa = \kappa_1 u^s$ is

$$N(u) = N_0 \exp\left\{-\frac{v^2}{\kappa_1\omega(1-s)\sin\theta}\left[(u_s^{1-s} - u^{1-s}) + \frac{1-s}{3-s}(u_0^{3-s} - u^{3-s})\right]\right\} \tag{5.18}$$

for $s \neq 1$, and

$$N(u) = N_0 \left(\frac{u}{u_0}\right)^{v^2\kappa_1 \sin\theta} \exp \frac{v^2(u^2 - u_0^2)}{2\kappa_1\omega \sin \theta} \tag{5.19}$$

for $s = 1$, subject to the boundary condition that $N = N_0$ at $u = u_0 \equiv (R\omega/v) \sin \theta$. The form $\kappa_1 u^s$ is probably not of much physical interest close to the Sun ($u < 1$) under these stationary conditions because the vanishing diffusion merely predicts vanishing particle density there.

It is interesting to note that if κ were of the form $1 + u^2$, then the Fokker–Planck equation reduces to the form for isotropic diffusion. In this case the particles are still

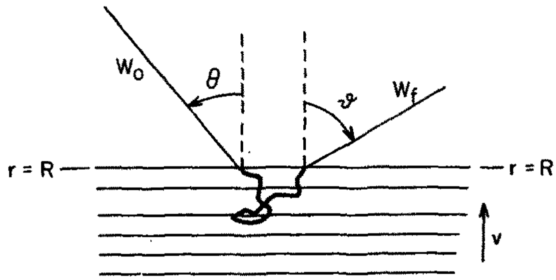

FIG. 11. THE GEOMETRY OF THE ENTRY AND EXIT OF AN ENERGETIC PARTICLE FROM INTERSTELLAR SPACE INTO THE SOLAR WIND. THE HORIZONTAL LINES SYMBOLIZE THE OUTWARD MOVING IRREGULARITIES WHICH VANISH UPON REACHING THE BOUNDARY $r = R$.

constrained to move along the magnetic lines of force, but the effect of the spiral on their radial motion is exactly compensated by the increase of κ with distance. The radial motion has the same dependence upon r and t as for the isotropic case.

APPENDIX 6

Fermi acceleration in the solar wind

Consider the energy transferred to a cosmic ray particle which comes head on into the solar wind from interstellar space and returns again to interstellar space. Neglect the adiabatic cooling of the particle while in the solar wind. It is perhaps easiest to think of the magnetic fields carried in the solar wind (and from which the particles scatter) as ranks of soldiers marching steadily outward with velocity v and dissolving into nothing as they reach $r = R$. This is illustrated schematically in Fig. 11, along with the trajectory of an incoming and outgoing cosmic ray particle. The particle approaches the outer boundary of the wind with a velocity w_0 at an angle θ from the normal. It interacts with the wind for a greater or lesser period of time, during which period it may be thought of as moving in the frame of reference of the wind, and subsequently escapes back into interstellar space with a velocity w_f at an angle ϑ from the normal. To compute w_f in terms of w_0, let w represent the constant speed of the particle in the frame of reference moving with the wind. Since w is equal to the initial velocity in the frame of reference moving with the wind, it is easy to show that, for the nonrelativistic case,

$$w^2 = w_0^2 + v^2 + 2vw_0 \cos \theta$$

The speed w is also equal to the final speed of the particle in the frame of reference moving with the wind, so that

$$w^2 = w_f^2 + v^2 - 2vw_f \cos \vartheta$$

Thus in the fixed frame of reference, in which w_0 and w_f are measured, the change in the square of the velocity upon entering the wind is

$$(\Delta w^2)_0 = w^2 - w_0^2 = 2vw_0 \cos\theta + O(v^2)$$

and upon leaving

$$(\Delta w^2)_f = 2vw_0 \cos\vartheta + O(v^2)$$

If it is assumed that the velocity distribution before entering and after leaving is isotropic, then the rate at which kinetic energy is transferred to the cosmic ray particles each of mass M, is

$$P_1 = \frac{4\pi R^2 \int_0^{\pi/2} d\theta \sin\theta \times N_0 w \cos\theta \times \frac{1}{2}M(\Delta w^2)_0}{\int_0^{\pi} d\theta \sin\theta} = \frac{4\pi R^3}{3}\frac{N_0 v T_0}{R}$$

upon entering the solar wind, with a similar result for the rate of energy transfer P_2 upon leaving. Hence

$$P' \equiv P_1 + P_2 = \frac{8\pi R^3}{3}\frac{N_0 v T_0}{R}$$

Now while a cosmic ray particle is knocking about among the magnetic irregularities in the solar wind, it may be accelerated by the Fermi mechanism because the magnetic irregularities presumably have some small random Alfvén velocity v_a relative to the wind. The energy gain of the particle with velocity w and energy T is $O(Tv_a^2/w^2)$ per collision[29]. If the mean free path between such collisions is L, then in a time t, one expects wt/L collisions and a total fractional energy gain

$$\frac{\Delta T}{T} = O\left(\frac{v_a^2 t}{wL}\right)$$

The Alfvén speed v_a at the orbit of Earth is typically 50 km/sec (5×10^{-5} G and 5 hydrogen atoms/cm^3) so that with $w = 3 \times 10^{10}$ cm/sec and $L = 10^7$ km we have $\Delta T/T = 10^{-3}$ in a typical time of 10^6 sec. This suggests that Fermi acceleration of cosmic ray particles is negligible while random walking in the magnetic irregularities in the solar wind.

Fermi[30] speculated that more efficient acceleration might arise if the magnetic field contained sharp kinks. Such a mechanism was demonstrated[31] using magnetic irregularities with sharp crests, in which case the energy increase of a particle is $O(Tv_a/w)$, so that after a time t

$$\frac{\Delta T}{T} = O\left(\frac{v_a t}{L}\right) \gg 1$$

This mechanism would lead to large, instead of negligible, changes in the energy of fast particles in interplanetary space, with extremely interesting consequences. However, we see no conclusive evidence in the magnetic records, such as Fig. 1, for the necessary sharp-crested hydromagnetic waves. So for the present, the conservative assumption is that Fermi acceleration of energetic particles is not an important phenomenon throughout interplanetary space.